Four Condorcet-Hare Hybrid Methods for Single-Winner Elections

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Abstract

This paper examines four single-winner election methods, denoted here as Woodall, Benham, Smith-AV, and Tideman, that all make use of both Condorcet's pairwise comparison principle and Hare's elimination and reallocation principle used in the alternative vote. These methods have many significant properties in common, including Smith efficiency and relatively strong resistance to strategic manipulation, though they differ with regard to the minor properties of 'Smith-IIA' and 'mono-add-plump'.

1 Introduction

The concept of majority rule is trickier than most people realize. When there are only two candidates in an election, then its meaning is quite clear: it tells us that the candidate with the most votes is elected. However, when there are more than two candidates, and no single candidate is the first choice of a majority, the meaning is no longer obvious.

The Condorcet principle[1] offers a plausible guideline for the meaning of majority rule in multi-candidate elections: if voters rank candidates in order of preference, and these rankings indicate that there is a candidate who would win a majority of votes in a one-on-one race against any other candidate on the ballot (a Condorcet winner), then we may interpret 'majority rule' as requiring his election.

[1] Condorcet (1785) defines this principle.

The weakness of this guideline is that it does not specify what majority rule requires when there is no Condorcet winner. For these situations, the Smith set provides a useful generalization of the Condorcet winner concept. The Smith set is the smallest set S such that any candidate in S would win a one-on-one race against any candidate not in S. Thus the Smith principle, which requires voting rules to select winning candidates from the Smith set, is an extension of the Condorcet principle that is applicable to all election outcomes.[2] For example, suppose that A is preferred by a majority to B, B is preferred by a majority to C, C is preferred by a majority to A, and all three of these candidates are preferred by majorities to D. In this case, electing A, B, or C is consistent with the majority rule guideline provided by the Smith principle, but electing D is not.

Several election methods have been proposed that satisfy the Smith principle. Among them are ranked pairs,[3] beatpath,[4] river,[5] Kemeny,[6] Nanson,[7] and Copeland.[8] However, the four methods on which this paper focuses possess another property, in addition to Smith efficiency, that makes them particularly interesting: they appear to be unusually resistant to strategic manipulation. Therefore, if a society wishes to choose among multiple options by majority rule given one balloting, and if it wishes to minimize the probability that

[2] Smith (1973) refers to his idea as a generalization of Condorcet consistency.
[3] Defined in Tideman (1987).
[4] Defined in Schulze (2003).
[5] Defined in Heitzig (2004).
[6] Defined in Kemeny (1959).
[7] See Tideman (2006), page 201–203.
[8] Defined in Copeland (1951).

voters will have an incentive to behave strategically, these methods are worthy of strong consideration.

These four methods also share the characteristic of employing the 'Hare principle', that is, the principle of eliminating the candidate with the fewest first-choice votes and reallocating those votes to other candidates.[9]

I will use the names Woodall, Benham, Smith-AV, and Tideman to refer to these rules, as they do not have standard names. They are deeply similar to one another and will choose the same winner in the vast majority of cases, but they are not identical. The purpose of this paper is to provide a solid understanding of how these methods work, how they differ from one another, and how they compare to other single-winner methods.

2 Preliminary Definitions

Assume that there are C candidates and V voters. Let τ be a tiebreaking vector that gives a unique score $\tau_c \in (0,1)$ to each candidate $c = 1, \dots, C$; τ can be random, predetermined, or determined by a tie-breaking ranking of candidates.[10] Let E be a vector of candidate eliminations, such that E_c is initially set to zero for each candidate $c = 1, \dots, C$. Let w denote the winning candidate. Let U_{vc} be the utility of voter v for candidate c. Let R_{vc} be the ranking that voter v gives to candidate c (such that lower-numbered rankings are better). All voting methods described in this paper, with the exception of approval voting and range voting, begin with the voters ranking the candidates in order of preference.

Pairwise comparison: An imaginary head-to-head contest between two candidates, in which each voter is assumed to vote for the candidate whom he gives a better ranking to. Formally, let $P_{xy} = \sum_{v=1}^{V} 1\{R_{vx} < R_{vy}\}$ be the number of voters who rank candidate x ahead of candidate y. If $P_{xy} > P_{yx}$, then x pairwise-beats y.

Condorcet winner: A candidate who wins all of his pairwise comparisons. Formally, x is a Condorcet winner if and only if $P_{xy} > P_{yx}, \forall y \neq x$.

Condorcet method: Any single-winner voting rule that always elects the Condorcet winner when one exists.

Majority rule cycle: A situation in which each of the candidates suffers at least one pairwise defeat, so that there is no Condorcet winner. Formally, $\forall x, \exists y\colon P_{yx} > P_{xy}$.

The Alternative Vote (AV):[11] The candidate with the fewest first choice votes (ballots ranking the candidate above all others in the race) is eliminated. The process is repeated until only one candidate remains.

Formally, in each round $r = 1, \dots, C-1$, we perform the following operations:

$$I_{vc} = 1\{[E_c = 0] \wedge [R_{vc} < R_{vc'}, \forall c'\colon (E_{c'} = 0 \wedge c' \neq c)]\}, \forall v, c.$$

$$F_c = \sum_{v=1}^{V} I_{vc} + \tau_c + E_c, \forall c.$$

$$z = \operatorname{argmin}(F).\ E_z = \infty.\ \Omega_z = r.$$

After round $C-1$, $w = \operatorname{argmin}(E)$, and $\Omega_w = C$.

Here, I is a V by C matrix indicating individual voters' top choices. F is a length-C vector of the candidates' first choice vote totals, which incorporates the unique fractional values in the tiebreaking vector τ in order to ensure that there will not be a tie for plurality loser. Infinity can be added to the F values of eliminated candidates to prevent them from being identified as the plurality loser in subsequent rounds. The vector Ω gives an 'elimination score' for each candidate, which will be used by the Woodall method.

Smith set:[12] Or, the 'minimal dominant set'. The smallest set of candidates such that every

[9] Thomas Hare offered the first voting procedure that included the iterative transfer of votes from plurality losers to candidates ranked next on ballots. See Hoag and Hallett (1926, 162–95). The first person to apply the 'Hare principle' to the election of a single candidate was Robert Ware, in 1871. See Reilly (2001, 33–34).

[10] See Zavist and Tideman (1989).

[11] Also known as instant runoff voting (IRV) and as the Hare method, the alternative vote (AV) is the application of proportional representation by the single transferable vote (STV) to the case of electing one candidate.

[12] This is so named because of Smith (1973). Schwartz (1986) refers to the Smith set as the

candidate inside the set is preferred by some majority of the voters to every candidate outside the set. When there is a Condorcet winner, it is the only member of the Smith set. Formally, the Smith set is the set of candidates S such that these conditions hold:

$\forall x \in S, \forall y \notin S, P_{xy} > P_{yx}$.

$\nexists S' \subsetneq S: (\forall x \in S', \forall y \notin S', P_{xy} > P_{yx})$.

3 Method Definitions

Woodall method:[13] Score candidates according to their elimination scores, and choose the Smith set candidate with best score.

That is, define each candidate's elimination score as the round in which he is eliminated by AV. (The AV winner is not eliminated, so we set his score to C.) If the Smith set has only one member, then this is the Woodall winner; otherwise, the winner is the candidate from inside the Smith set who has the best elimination score.

Formally, we begin with the definitions of the AV method and Smith set as given above. Then, $\Upsilon_c = 1\{c \in S\} \cdot \Omega_c, \forall c$, and $w = \text{argmax}(\Upsilon)$.

Benham method:[14] Eliminate the plurality loser until there is a Condorcet winner.

That is, if there is a Condorcet winner, he is also the Woodall winner. Otherwise, the method eliminates the candidate with the fewest first-choice votes, and checks to see whether there is a candidate who beats all other non-eliminated candidates pairwise. This process repeats until there is such a candidate, who is then declared the winner.

Formally, in each round we determine whether

$\exists x: [(P_{xy} > P_{yx}, \forall y: E_y = 0) \wedge (E_x = 0)]$.

If so, then $w = x$, and the process stops. Otherwise, we perform these calculations:

$I_{vc} = 1\{[E_c = 0] \wedge [R_{vc} < R_{vc'}, \forall c'$:

$(E_{c'} = 0 \wedge c' \neq c)]\}, \forall v, c$.

$F_c = \sum_{v=1}^{V} I_{vc} + \tau_c + E_c, \forall c$.

$z = \text{argmin}(F)$. $E_z = \infty$.

Then, we proceed to the next round.

Smith-AV method:[15] Eliminate candidates not in the Smith set, and then conduct an AV tally among remaining candidates.

Tideman method:[16] Alternate between eliminating all candidates outside the Smith set, and eliminating the plurality loser, until one candidate remains.

That is, as in Smith-AV, we begin by eliminating all candidates outside the Smith set. If this leaves only one candidate (a Condorcet winner), then he is elected. Otherwise, we eliminate the candidate with the fewest first choice votes. Then, we recalculate the Smith set, and eliminate any candidates who were in it before but are no longer in it as a result of the plurality loser elimination. These two steps repeat until only one candidate (the winner) remains.

Formally, in stage 1, we define or re-define S according to the following conditions:

$\forall x \in S, \forall y$:
$(y \notin S \wedge E_y = 0), P_{xy} > P_{yx}. \forall x \in S, E_x = 0$.

$\nexists S' \subsetneq S$:
$[\forall x \in S', \forall y: (y \notin S' \wedge E_y = 0), P_{xy} > P_{yx}]$.

Then, we make the following adjustment to the E vector: $c \notin S \rightarrow E_c = \infty$.

In stage 2, we perform the following calculations:

GETCHA set, and also defines another set called the GOCHA set, which is now also known as the Schwartz set. The Schwartz set is the union of minimal undominated sets, where an undominated set is a set such that no member of the set is pairwise-defeated by a non-member. (This is equivalent to the Smith set in the absence of pairwise ties.) Though the methods defined in this paper are based on the Smith set, each has a potential Schwartz-set counterpart.

[13] Woodall (2003) defines this method (among many, many others), and refers to it as CNTT, AV, for 'Condorcet (net) top tier, alternative vote'.

[14] I'm not aware of any academic papers that define this method, but it was suggested to me by Chris Benham.

[15] Woodall (1997) lists this method under the heading 'naïve rules'. I refer to it as Smith-AV because it seems like the most obvious combination of the Smith set and AV.

[16] Tideman (2006) defines this method on page 232 and refers to it as alternative Smith.

$I_{vc} = 1\{[E_c = 0] \wedge$
$[R_{vc} < R_{vc'}, \forall c' : (E'_c = 0 \wedge c' \neq c)]\}, \forall v, c.$

$F_c = \sum_{v=1}^{V} I_{vc} + \tau_c + E_c, \forall c.$

$z = \mathrm{argmin}(F).\ E_z = \infty.$

Stages 1 and 2 alternate until S has only one member, i.e. $S = \{w\}$.

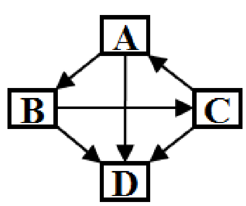
4 Examples

Examples 1 and 2 demonstrate how the four methods work, and prove that none of them are equivalent to any of the others. To help illustrate each calculation, I present the pairwise matrix, P, and a corresponding tournament diagram that uses arrows to represent pairwise defeats. I also present round-by-round tallies for the different methods, which show how many first choice votes each candidate holds at each stage of the count, along with the transfers of first choice votes from eliminated candidates.

Example 1: Woodall and Benham differ from Smith-AV and Tideman

6	DABC
5	BCAD
4	CABD

P	A	B	C	D
A		10	6	9
B	5		11	9
C	9	4		9
D	6	6	6	



	r1		r2	
A	0	X	-	
B	5		5	✔
C	4		4	X
D	6		6	X

Benham tally

	r1		r2		r3	
A	0	X	-		-	
B	5		5	+4	9	✔
C	4		4	X	-	
D	6		6		6	X

AV tally

	r1		r2		r3	
A	0	+6	6	+4	10	✔
B	5		5		5	X
C	4		4	X	-	
D	6	X	-		-	

Smith-AV or Tideman tally

Woodall: In an AV tally, A is eliminated first, followed by C and then D, leaving B as the winner. The Smith set is {A,B,C} Therefore, **B** is the Smith set candidate with the best AV score.

Benham: There is no Condorcet winner, so we eliminate A, who is the plurality loser. B is a Condorcet winner among the remaining candidates, so **B** wins.

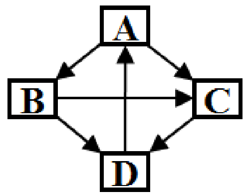
Smith-AV: D is not in the Smith set, so he is eliminated. C is eliminated in the first AV counting round, and B is eliminated in the second AV counting round, so **A** is the winner.

Tideman: This rule works the same as Smith-AV in this example, and thus elects **A**. In the last phase, B is eliminated because he is no longer in the Smith set rather than because he is the plurality loser, but with only two candidates remaining, these are equivalent.

Example 2: Benham and Tideman differ from Woodall and Smith-AV

4	ABCD
5	BDAC
6	CDAB

P	A	B	C	D
A		10	9	4
B	5		9	9
C	6	6		10
D	11	6	5	



	r1		r2		r3	
A	4		4	X	-	
B	5		5	+4	9	✔
C	6		6		6	X
D	0	X	-		-	

AV or Smith-AV tally

	r1		r2	
A	4		4	✔
B	5		5	X
C	6		6	X
D	0	X	-	

Benham or Tideman tally

Woodall: In an AV tally, D is eliminated first, followed by A, and then C, leaving B as the winner. Therefore, **B** is the Smith set candidate with the best AV score.

Benham: There is no Condorcet winner, so we eliminate D who is the plurality loser. A is the Condorcet winner among remaining candidates, so **A** wins.

Smith-AV: All candidates are in the Smith set, so we proceed to the AV tally. D has no first-choice votes, so he is eliminated in the first AV counting round. In the second AV round, A has 4 first choice votes, B has 5, and C has 6, so A is eliminated. In the third AV round, C is eliminated, and **B** wins.

Tideman: All candidates are in the Smith set. The plurality loser is D, so he is eliminated. Recalculating the Smith set, we find that A is now the Condorcet winner, so **A** wins.

5. Strategic Voting

There is no single, agreed way to measure vulnerability to strategic voting, but one approach is to simulate elections using a specified data-generating process, and then to determine the percentage of trials in which coalitional manipulation is possible in each method.[17] That is, in what percentage of trials does there exist a group of voters who all prefer another candidate to the sincere winner, and who can cause that candidate to win by changing their votes?

Here, I will present results arising from two data generating processes: a spatial model, and an impartial culture model. I recognize that this is not exhaustive, as there are an infinite number of possible data generating processes, but it will serve at least to give preliminary evidence, and to demonstrate some basic principles.[18]

The spatial voting model used here distributes both voters and candidates randomly in N-dimensional issue space, according to a multivariate normal distribution without covariance. Voters are then assumed to prefer candidates who are closer to them in this issue space. Formally,

$$L_{vn} \sim \mathcal{N}(0,1), \forall v, n.$$

$$\Lambda_{cn} \sim \mathcal{N}(0,1), \forall c, n.$$

$$U_{vc} = -\sqrt{\sum_{n=1}^{N} (L_{vn} - \Lambda_{cn})^2}, \forall v, c.$$

(The L and Λ matrices give the voter and candidate locations, respectively.)

The impartial culture model used here simply treats each voter's utility over each candidate as an independent draw from a uniform distribution, thus making each ranking equally probable, independent of other voters' rankings. Formally, $U_{vc} \sim \mathcal{U}(0,1), \forall v, c$.

In order to avoid massive computational cost, I make the restrictive assumption that all voters in the strategic coalition must cast the same ballot. Thus, I am not computing the

[17] For example, see Chamberlin (1985), Lepelley and Mbih (1994), Kim and Roush (1996), Favardin, Lepelley, and Serais (2002), Favardin and Lepelley (2006), Tideman (2006), and Green-Armytage (2011).

[18] Green-Armytage (2011) also uses the voter ratings of politicians in the American National Election Studies time series survey as a data generating process, and finds that it gives similar results to the models used here.

frequency with which manipulation is possible, but rather finding a lower-bound approximation.[19]

Tables 1 and 2 show the results of this analysis, given various specifications of the spatial model and the impartial culture model, respectively. I use 10,000 trials for each specification, which causes the margin of error to be .0098 or less,[20] with 95% confidence. In addition to applying the analysis to Woodall, Benham, Smith-AV, and Tideman, I apply it to AV, ranked pairs, beatpath, plurality,[21] minimax,[22] Borda,[23] approval voting,[24] and range voting.[25] Figures 1 and 2 illustrate a subset of these results. To make the graphs less convoluted, I allow Woodall to stand in for the other three Condorcet-Hare hybrids, I allow minimax to stand in for ranked pairs and beatpath, and I allow approval voting to stand in for range voting.

In every one of these specifications, the five methods that are least frequently manipulable are Woodall, Benham, Smith-AV, Tideman, and AV. Among these methods, AV is vulnerable with slightly greater frequency, but the difference tends to be very small. Likewise, there are some specifications in which Woodall and Benham outperform Smith-AV and Tideman, but their scores are usually extremely close or identical. Minimax, beatpath, and ranked pairs are all vulnerable with substantially greater frequency than these five, but they are all vulnerable with substantially lower frequency than plurality, which in turn is vulnerable with substantially lower frequency than Borda, approval, and range.

One notable feature of the spatial model is that vulnerability is substantially higher across the board when $N = 1$, and that it decreases rapidly as N increases. Given $N > 1$, the difference between the best five methods and the remaining methods is particularly striking. One notable feature of the impartial culture model is that although the probability that a method will be vulnerable to manipulation seems to converge to 100% as V becomes large for all of the other methods included here, it doesn't do so for AV and the Smith-AV hybrids.

Why are AV and the Condorcet-Hare hybrids vulnerable with lower frequency than the other methods? To give some intuition for this, it may be helpful to define two particular types of strategic voting: 'compromising' and 'burying'. Suppose that w is the sincere winner, and q is an alternative candidate whom strategic voters are seeking to elect instead. In this context, the **compromising strategy** would be their giving q a better ranking (or rating), and the **burying strategy** would be their giving w a worse ranking (or rating).[26] Together, these tactics seem to account for most strategic possibilities.[27]

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[19] Green-Armytage (2011) performs calculations that don't rely on this assumption, but these calculations are not applied to any Condorcet-Hare hybrid meth+ods. Doing so without massive computational cost presents a set of interesting programming challenges. Meanwhile, comparing the results from the two papers suggests that the assumption of uniform strategic coalitions has only a minor impact on the manipulability of most methods.

[20] A margin of error of ±.0098 is the upper bound, which applies when the true probability is exactly one half. I further reduce the random error in the *difference* between the scores that the various voting methods receive by using the same set of randomly generated elections for each method.

[21] I define the plurality winner as the candidate with the most first choice votes.

[22] The minimax winner is the Condorcet winner if one exists, or otherwise, the candidate whose worst loss is least bad. Formally:

$$\mathrm{M}_y = \max_{x=1}^{C} P_{xy} - \tau_y, \forall y = 1, \ldots, C.$$

$$w = \operatorname{argmin}(\mathrm{M}).$$

[23] The Borda winner is the candidate with the most points, if each first choice vote is worth C points, each second choice vote is worth $C - 1$ points, and so on. Equivalently, Borda can be calculated as follows:

$$\mathrm{B}_y = \sum_{x=1}^{C} P_{xy} - \tau_y, \forall y = 1, \ldots, C.$$

$$w = \operatorname{argmin}(\mathrm{B}).$$

[24] Each voter can give each candidate either one point or zero points. The winner is the candidate with the most points.

[25] Each voter can give each candidate any number of points in a specified range, e.g. 0 to 100. The winner is the candidate with the most points.

[26] The terms 'compromising' and 'burying' were used by Blake Cretney in the currently-defunct web site condorcet.org.

[27] This is somewhat intuitive, and supporting evidence is given in Green-Armytage (2011).

Table 1: Strategic voting, spatial model

V	*N*	*C*	Woodall	Benham	Smith-AV	Tideman	AV	minimax	beatpath	ranked pairs	plurality	Borda	approval	range
99	1	3	.140	.140	.140	.140	.140	.152	.152	.152	.282	.395	.549	.594
99	1	4	.325	.325	.335	.335	.325	.351	.359	.359	.549	.825	.798	.865
99	1	5	.487	.487	.512	.512	.487	.550	.560	.560	.732	.980	.912	.960
99	1	6	.622	.622	.660	.660	.622	.694	.707	.707	.844	.998	.960	.988
99	2	3	.038	.038	.038	.038	.045	.191	.189	.189	.229	.424	.500	.500
99	2	4	.104	.104	.107	.107	.119	.358	.359	.359	.492	.734	.731	.785
99	2	5	.186	.186	.194	.194	.209	.490	.492	.492	.693	.900	.840	.905
99	2	6	.262	.262	.279	.279	.287	.599	.601	.601	.825	.964	.903	.961
99	3	3	.019	.019	.019	.019	.026	.192	.192	.192	.212	.426	.470	.468
99	3	4	.044	.044	.044	.044	.059	.333	.333	.333	.440	.707	.684	.733
99	3	5	.077	.077	.080	.080	.100	.431	.431	.431	.617	.854	.796	.861
99	3	6	.116	.116	.122	.123	.146	.520	.521	.521	.765	.927	.871	.933
99	4	3	.013	.013	.013	.013	.020	.198	.198	.198	.210	.426	.463	.457
99	4	4	.031	.031	.032	.032	.044	.321	.321	.321	.419	.697	.668	.710
99	4	5	.048	.048	.049	.049	.068	.413	.413	.413	.599	.835	.779	.848
99	4	6	.065	.065	.068	.068	.091	.478	.480	.480	.726	.908	.854	.915
99	16	3	.002	.002	.002	.002	.006	.186	.183	.183	.187	.416	.432	.431
99	16	4	.007	.007	.007	.007	.015	.290	.291	.291	.369	.653	.629	.658
99	16	5	.010	.010	.010	.010	.020	.350	.350	.350	.497	.770	.733	.772
99	16	6	.014	.014	.014	.014	.028	.399	.398	.398	.601	.843	.807	.845

Table 2: Strategic voting, impartial culture model

V	*C*	Woodall	Benham	Smith-AV	Tideman	AV	minimax	beatpath	ranked pairs	plurality	Borda	approval	range
9	3	.101	.101	.101	.101	.119	.357	.342	.342	.389	.560	.599	.606
9	4	.213	.213	.216	.216	.235	.557	.557	.557	.625	.794	.775	.829
9	5	.307	.307	.313	.314	.333	.682	.694	.694	.763	.897	.858	.910
9	6	.389	.389	.402	.403	.419	.763	.781	.781	.847	.943	.911	.952
29	3	.099	.099	.099	.099	.126	.681	.676	.676	.694	.816	.837	.843
29	4	.188	.188	.188	.188	.231	.846	.845	.845	.921	.965	.952	.976
29	5	.282	.282	.285	.285	.335	.912	.914	.914	.981	.989	.981	.996
29	6	.355	.355	.362	.362	.415	.948	.948	.948	.995	.995	.993	.998
99	3	.088	.088	.088	.088	.123	.951	.952	.952	.951	.989	.986	.990
99	4	.180	.180	.180	.180	.241	.987	.987	.987	.999	.998	.999	1.000
99	5	.255	.255	.255	.255	.327	.995	.995	.995	1.000	.993	1.000	1.000
99	6	.312	.312	.312	.312	.405	.998	.998	.998	1.000	.979	1.000	1.000

Figure 1: Strategic voting, spatial model, N=4, V=99

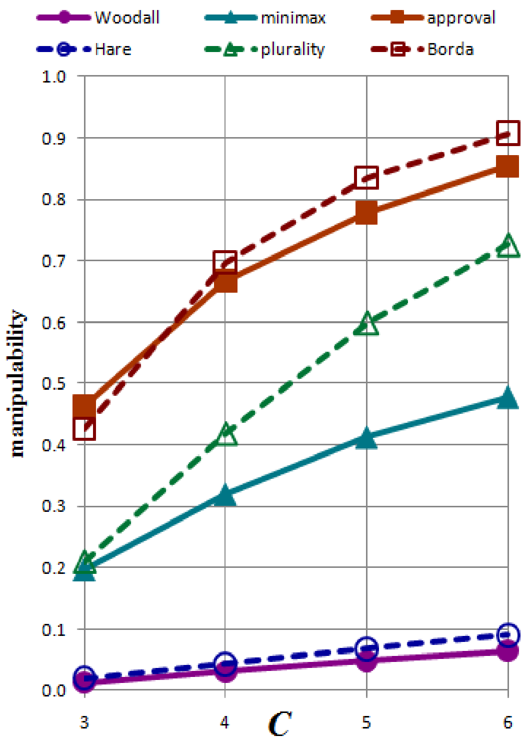
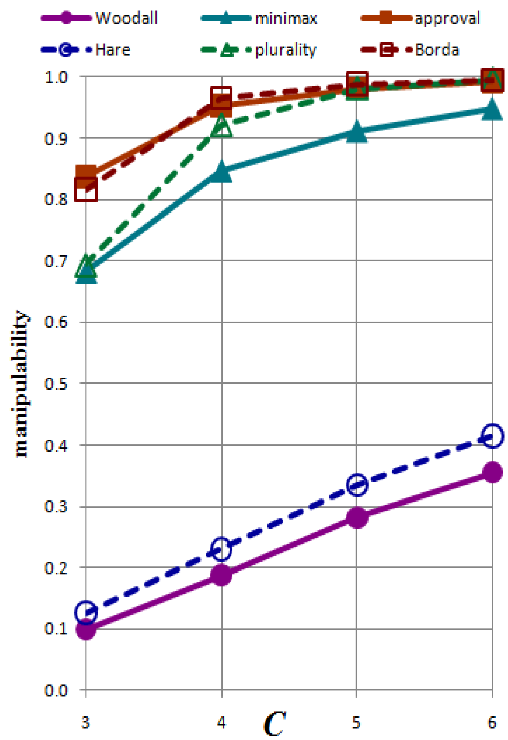


Figure 2: Strategic voting, impartial culture model, V=29



AV is immune to the burying strategy, and it is only vulnerable to the compromising strategy in relatively rare situations, such as when the AV winner and Condorcet winner are different, or when there is a majority rule cycle. The Condorcet-Hare hybrids are strictly less vulnerable to compromising, in that they are only vulnerable when there is a majority rule cycle. All Condorcet-efficient methods are vulnerable to burying,[28] but this vulnerability seems to be substantially less frequent in the Condorcet-Hare hybrids than in most other Condorcet methods. The reason for this is that voters who prefer q to w will already have ranked q ahead of w, so that further burying w will not affect w's plurality score unless q has already been eliminated. Burying w can create a cycle with q and some other candidate or candidates, but unless w already happens to be the plurality loser among the candidates in this cycle, the strategy is unlikely to actually elect q.

[28] Woodall (1997) demonstrates that Condorcet is incompatible with the properties of 'later-no-help' and 'later-no-harm', which is a nearly equivalent statement.

6 Strategic Nomination

A comparable method can be applied to measuring the frequency of incentives for strategic nomination, which I define here as non-winning candidates entering or leaving the race in order to change the results to ones they prefer.[29] For example, suppose that A wins given the set of candidates {A,B,C}, but B wins given the set {A,B}, and candidate C prefers B to A. In this case, candidate C has an incentive for **strategic exit**. Alternatively, suppose that X wins given the set of candidates {X,Y}, but Y wins given the set {X,Y,Z}, and Z prefers Y to X. In this case, candidate Z has an incentive for **strategic entry**.

I use only the spatial model for my strategic nomination analysis here, because it provides the more straightforward method of determining candidates' preferences over other candidates; that is, it is natural to imagine that candidates prefer other candidates who are closer to them in the issue space. Formally,

[29] This analysis follows Green-Armytage (2011).

$\Psi_{xy} = -\sqrt{\sum_{n=1}^{N}(\Lambda_{xn} - \Lambda_{yn})^2}$ gives the utility of candidate x if candidate y wins (and vice versa).

Aside from V and N, the parameters of the model are CI and CO, which represent the number of candidates who are initially in the race (but who have the ability to exit), and the number of candidates who are initially out of the race (but who have the ability to enter).

I exclude approval and range from this analysis, because any effects that show up will only be an artefact of the way that utilities are transformed into approval votes and range scores, respectively. If this transformation is independent of which candidates are actually running, then nomination vulnerability is always zero.

Tables 3 and 4, and figures 3 and 4, present the result of some strategic nomination simulations, once again with 10,000 trials per specification.

The most salient result here is that all of the Condorcet methods are only slightly vulnerable to both strategic exit and strategic entry, while other methods are more vulnerable. Plurality is highly vulnerable to strategic exit; presumably, this helps to explain the common practice of holding party primaries so that candidates with similar ideologies don't get in each others' way. AV is substantially vulnerable to strategic exit as well, especially when C is large. Borda is the most vulnerable to strategic entry.

Condorcet methods are vulnerable to strategic exit only if there is a majority rule cycle among the candidates who are in the race; if there is a Condorcet winner to begin with, he will remain the Condorcet winner after the deletion of any other candidate.[30] Likewise, they are vulnerable to strategic entry only if there is a cycle when the newly-entered candidate is included. In the spatial model, majority rule cycles are rare, so Condorcet methods are rarely vulnerable to strategic nomination.

[30] Note that the existence of a cycle doesn't necessarily imply an incentive for strategic exit, though it does imply an incentive for strategic voting.

7 Mathematical Properties

We will see in this section that the four Condorcet-Hare hybrids are similar enough to have the same status with respect to most mathematical properties. Like all other Condorcet-efficient rules, they lack participation,[31] and like AV, they lack monotonicity as well. Meanwhile, they possess Smith consistency, along with properties that are implied by this, such as Condorcet, Condorcet loser,[32] strict majority,[33] and mutual majority.[34]

While thus sharing many properties, these methods can nevertheless be distinguished on the basis of lesser-known (and arguably less significant) properties. For example, Smith-AV and Tideman have a property called 'Smith-IIA', but lack two properties called 'mono-add-plump' and 'mono-append', whereas for Woodall and Benham, the opposite is true.

7.1. Monotonicity

Definition: If x is not the winner, then changing ballots by giving x an inferior ranking will never change the winner to x. (Conversely, if x is the winner, then changing ballots by giving x a superior ranking will never cause x to lose.)

Example 3: Woodall, Benham, Smith-AV, Tideman, and AV all lack monotonicity

7	ABC
10	BCA
6	CAB

Given any of the five systems, the initial winner is A, but if two of the BCA voters change their votes to CBA, the winner will change to B.

[31] Moulin (1988) demonstrates that no method can simultaneously possess Condorcet consistency and the participation property.

[32] A Condorcet loser is a candidate who loses all pairwise comparisons. The Condorcet loser property states that such a candidate never wins.

[33] This property states that if candidate x is ranked first by a majority of voters, then x is elected.

[34] This property states that if there is a set of candidates such that a cohesive majority of voters ranks all members in the set ahead of all members outside the set, then the winner is a member of the set.

Table 3: Strategic exit

V	*S*	*CO*	*CI*	Woodall	Benham	Smith-AV	Tideman	AV	minimax	beatpath	ranked pairs	plurality	Borda
99	4	0	3	.002	.002	.002	.002	.015	.001	.001	.001	.091	.006
99	4	0	5	.007	.007	.007	.007	.060	.004	.003	.004	.251	.013
99	4	0	7	.010	.010	.010	.010	.104	.006	.005	.005	.356	.017
99	4	0	9	.015	.015	.015	.015	.151	.008	.008	.009	.434	.021
99	4	0	11	.018	.017	.018	.018	.193	.010	.009	.010	.490	.018
99	4	0	13	.022	.021	.022	.021	.245	.012	.012	.012	.526	.022
99	4	0	15	.025	.025	.025	.025	.298	.013	.013	.014	.546	.026
99	4	0	19	.033	.030	.032	.030	.389	.015	.013	.014	.588	.027
99	4	0	23	.036	.033	.035	.034	.468	.017	.016	.016	.605	.026
99	4	0	27	.041	.039	.040	.038	.533	.018	.018	.018	.627	.022

Table 4: Strategic entry

V	*S*	*CO*	*CI*	Woodall	Benham	Smith-AV	Tideman	AV	minimax	beatpath	ranked pairs	plurality	Borda
99	4	1	2	.000	.000	.000	.000	.000	.001	.001	.001	.001	.015
99	4	2	2	.000	.000	.000	.000	.000	.001	.001	.001	.004	.029
99	4	3	2	.000	.000	.000	.000	.001	.002	.002	.002	.003	.038
99	4	5	2	.000	.000	.000	.000	.001	.002	.002	.002	.008	.059
99	4	7	2	.001	.001	.001	.001	.002	.002	.003	.003	.009	.065
99	4	9	2	.000	.000	.000	.000	.002	.003	.003	.003	.009	.076
99	4	11	2	.001	.001	.001	.001	.002	.005	.005	.005	.013	.094
99	4	13	2	.001	.001	.001	.001	.002	.005	.005	.005	.016	.103
99	4	15	2	.001	.001	.001	.001	.002	.005	.005	.005	.018	.101
99	4	19	2	.002	.002	.002	.002	.004	.006	.007	.007	.020	.113
99	4	23	2	.001	.001	.001	.001	.004	.006	.006	.006	.024	.118
99	4	27	2	.003	.003	.003	.003	.005	.009	.008	.008	.027	.129

Figure 3: Strategic exit, N=4, V=99

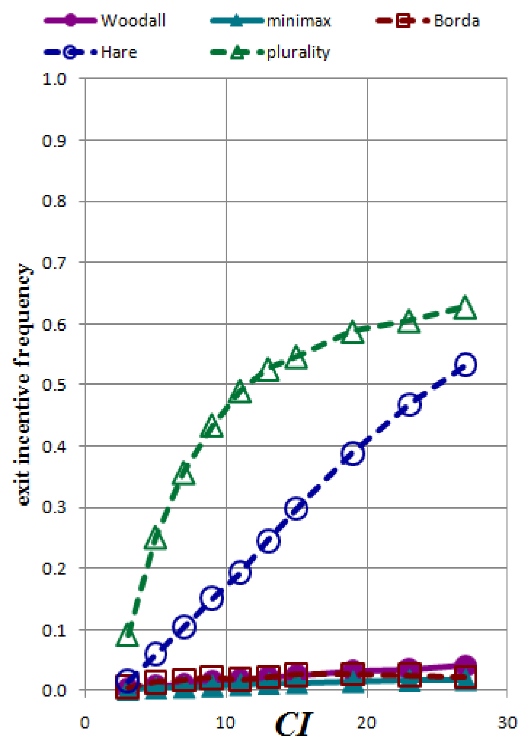
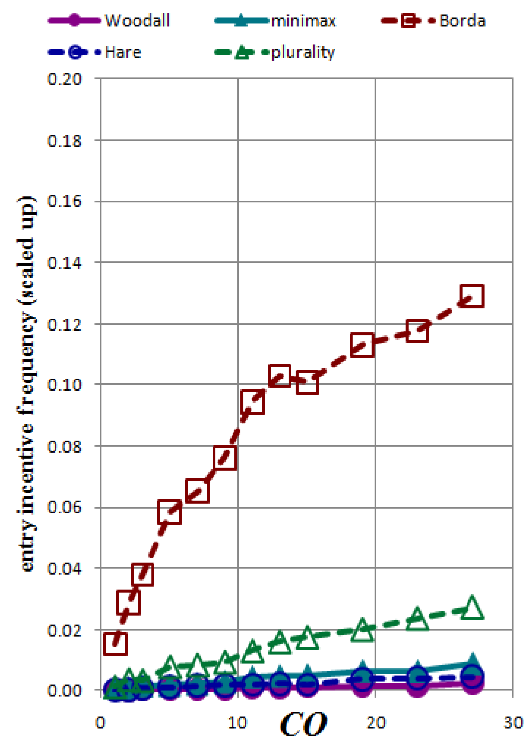


Figure 4: Strategic entry, CI=2, N=4, V=99



7.2. Participation

Definition: If the initial winner is x, and an extra vote is added that ranks x ahead of y, it will never change the winner to y.

Discussion: To lack this property is also known as the no-show paradox. This property is closely related to another property, known variously as consistency,[35] separability,[36] and combinativity,[37] which states that if x is the winner according to each of two separate sets of ballots, then x will be the winner when the sets are combined.

Example 4: Woodall, Benham, Smith-AV, Tideman, and AV all lack participation

4	ABC
5	BCA
6	CAB

Assume that ties are broken lexicographically. Given any of the four systems, the initial winner is B, but adding another ABC voter changes the winner to C.

7.3. Mono-add-plump[38]

Definition: If x is the winner, and one or more ballots are added that rank x first, and indicate no further rankings, then x will necessarily remain the winner.

Discussion: This property can be thought of as a weaker version of the participation property or the consistency property.

Example 5: Smith-AV and Tideman lack mono-add-plump

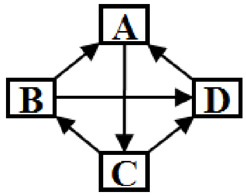
8	ACBD
3	BACD
7	CBDA
5	DBAC

[35] In Young (1975).
[36] In Smith (1973).
[37] In Tideman (2006).

[38] This property is defined in Woodall (1996), along with mono-append below. I credit Chris Benham with pointing out that these properties provide a distinction between Woodall and Benham on one hand, and Smith-AV and Tideman on the other.

Tallies for Example 5 without and with added ballots

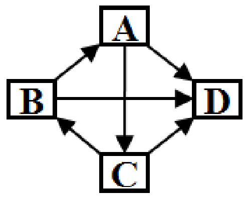
P	A	B	C	D
A		8	16	11
B	15		8	18
C	7	15		18
D	12	5	5	



	round 1		round 2		round 3	
A	8	+3	11	+5	16	✔
B	3	X	-		-	
C	7		7		7	X
D	5		5	X	-	

Smith-AV or Tideman tally

P	A	B	C	D
A		10	18	13
B	15		8	18
C	7	15		18
D	12	5	5	



	round 1		round 2		round 3	
A	10		10		10	X
B	3	+5	8	+7	17	✔
C	7		7	X	-	
D	5	X	-		-	

Smith-AV or Tideman tally

Given these ballots, A will win under both Smith-AV and Tideman. However, adding two voters who only indicate a first preference for A will change the winner to B. (Adding the A-only votes removes D from the Smith set, which in turn strengthens B.) The tallies are presented above, first without the extra ballots, and then with them.

Proposition 1: Woodall possesses mono-add-plump

Proof:

1. Suppose that with the original set of ballots, candidate *x* wins in round *r*. That is, if the Smith set has any members other than *x*, they are eliminated before round *r* in the AV count.
2. Adding *x*-only ballots will not affect the order in which candidates are eliminated in any round before *r*.
3. Adding *x*-only ballots will not remove *x* from the Smith set.
4. Adding *x*-only ballots will not add new candidates to the Smith set.
5. In view of 2–4, adding *x*-only ballots can’t prevent candidate *x* from winning in round *r*. ■

Proposition 2: Benham possesses mono-add-plump

Proof:

1. Suppose that with the original set of ballots, candidate *x* wins in round *r*. That is, as of round *r*, *x* is a Condorcet winner among the remaining candidates.
2. Adding *x*-only ballots will not affect the order in which candidates are eliminated in any round before *r*. Therefore, the set of non-eliminated candidates in round *r* will not be changed.
3. If *x* is a Condorcet winner among a given set of candidates, adding *x*-only ballots will not change this.
4. In view of 2 and 3, adding *x*-only ballots can’t prevent candidate *x* from winning in round *r*. ■

Table 5: overall summary

	Woodall	Benham	Smith-AV	Tideman	AV	beatpath, ranked pairs	minimax	plurality	Borda	approval, range

Smith	✔	✔	✔	✔	X	✔	X	X	X	X
HRSV	✔	✔	✔	✔	✔	X	X	X	X	X
HRSN	✔	✔	✔	✔	X	✔	✔	X	X	✔
monotonicity	X	X	X	X	X	✔	✔	✔	✔	✔
participation	X	X	X	X	X	X	X	✔	✔	✔
Condorcet	✔	✔	✔	✔	X	✔	✔	X	X	X
Condorcet loser	✔	✔	✔	✔	✔	✔	✔	X	✔	X
strict majority	✔	✔	✔	✔	✔	✔	✔	✔	X	X
mutual majority	✔	✔	✔	✔	✔	✔	X	X	X	X
Smith-IIA	X	X	✔	✔	X	✔	X	X	X	X
MAP/MA	✔	✔	X	X	✔	✔	✔	✔	✔	✔

7.4. Mono-append

Definition: If *x* is the winner, and one or more ballots that previously left *x* unranked are changed only in that *x* is added to the ballot after the last ranked candidate, then *x* will necessarily remain the winner.

Discussion: This property is fairly similar to mono-add-plump.

Example 6: Smith-AV and Tideman lack mono-append

10 ACBD
3 B
7 CBDA
5 DBAC

With these ballots, A will win both Smith-AV and Tideman. However, changing the 3 B votes to BA votes will change the winner to B. (Again, this strengthens B by removing D from the Smith set.)

It is fairly easy to see that Woodall and Benham possess mono-append, following logic similar to that of the proofs of propositions 1 and 2 above.

7.5. Smith-IIA[39]

Definition: Removing a candidate from the ballot who is not a member of the Smith set will not change the result of the election. (The 'IIA' here stands for 'independence of irrelevant alternatives'.)

Example 1 above shows that Woodall and Benham lack this property. That is, removing D will change the winner from B to A.

It is easy to see that Smith-AV and Tideman both possess this property, because both methods begin by eliminating candidates outside the Smith set.

8 Conclusion

Table 5 summarizes the results from sections 5–7. HRSV and HRSN are abbreviations for 'highly resistant to strategic voting', and 'highly resistant to strategic nomination'. (Of course, reducing the simulation results to a binary score requires the imposition of a somewhat arbitrary cut-off, but in general, the methods deemed 'highly resistant' in each category perform substantially better than the others.) MAP/MA is an abbreviation for mono-add-plump and mono-append.

Woodall, Benham, Smith-AV, and Tideman possess Smith consistency (and therefore the Condorcet, Condorcet loser, strict majority, and mutual majority properties), and offer relatively few opportunities for strategic voting and strategic nomination; I suggest that this combination of properties could be valuable if applied to single-winner public elections. I don't conclude that any of these methods is unambiguously better than the others; rather, I

[39] Defined in Schulze (2003).

leave it to the reader to decide which one he or she prefers.

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