QPQ, a quota-preferential STV-like election rule

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1 Introduction

Olli Salmi, in a posting to an Election Methods list [6], has suggested a new quota-preferential election rule, which is developed slightly further in this article, and which is remarkably similar to the Single Transferable Vote (STV) in its effects. I shall call it QPQ, for Quota-Preferential by Quotient. Both in its properties and in the results it gives, it seems to be more like Meek's version of STV [2] than the traditional version [3]. This is surprising since: (i) in marked contrast with STV, the quota in QPQ is used only as a criterion for election, and not in the transfer of surplus votes; (ii) QPQ, unlike Meek's method, involves no iterative processes, and so the votes can be counted by hand; and (iii) QPQ derives from the European continental tradition of party list systems (specifically, d'Hondt's rule), which is usually regarded as quite different from STV. I do not imagine that anyone who is already using STV will see any reason to switch to QPQ; but people who are already using d'Hondt's rule may feel that QPQ is a natural progression of it, and so more acceptable than STV.

D'Hondt's rule for allocating seats to parties was proposed by the Belgian lawyer Victor d'Hondt [1] in 1882. The seats are allocated to the parties one by one. At each stage, a party with v votes and (currently) s seats is assigned the quotient $v/(1+s)$, and the next seat is allocated to the party with the largest quotient. This continues until all seats have been filled.

Many variations of this rule were subsequently proposed, in which the divisor $1+s$ is replaced by some other function of s. However, the next contribution of relevance to us is an adaptation of d'Hondt's rule to work with STV-type preferential ballots. This adaptation has been part of Sweden's Elections Act for many years; we will call it the *d'Hondt–Phragmén method*, since it is based on a method proposed by the Swedish mathematician Lars Edvard Phragmén [4, 5] in 1895. The seats are again allocated one by one, only this time to candidates rather than parties; at each stage, the next seat is allocated to the candidate with the largest quotient (calculated as explained below). In the event that the voters effectively vote for disjoint party lists (e.g., if every ballot is marked for *abcd*, *efg* or *hijkl*), then the d'Hondt–Phragmén method gives exactly the same result as d'Hondt's rule. However, it was introduced in the Swedish Elections Act as a means of allocating seats *within* a party, at a time when voters were allowed to express a choice of candidates within the party. It does not guarantee to represent minorities proportionally.

Salmi's contribution has been to introduce a quota into Phragmén's method. In this version, which he calls the *d'Hondt–Phragmén method with quota*, the candidate with the largest quotient will get the next seat if, and only if, this quotient is larger than the quota; otherwise, the candidate with the smallest quotient is excluded, and the quotients are recalculated. In this respect it is like STV. However, unlike in STV, this is the only way in which the quota is used; it is not used in transferring votes. QPQ, as described here, differs from Salmi's original version only in that the quota is defined slightly differently, and the count is preferably restarted after every exclusion.

Both the d'Hondt–Phragmén method (with or without quota), and QPQ, can be described in terms of groups of voters rather than individuals, and this is naturally how one thinks when processing piles of ballots by hand. But it seems to me that they are easier to understand when rewritten in terms of individual ballots rather than groups, and they are described here in this form. From now on, s denotes the total number of seats to be filled.

2 The details of QPQ

2.1. The count is divided into a sequence of stages. At the start of each stage, each candidate is in one of three states, designated as *elected*, *excluded* and *hopeful*.

At the start of the first stage, every candidate is hopeful. In each stage, either one hopeful candidate is reclassified as elected, or one hopeful candidate is reclassified as excluded.

2.2. At the start of each stage, each ballot is deemed to have elected some fractional number of candidates, in such a way that the sum of these fractional numbers over all ballots is equal to the number of candidates who are currently classed as elected. At the start of the first stage, every ballot has elected 0 candidates.

2.3. At the start of each stage, the quotients of all the hopeful candidates are calculated, as follows. The ballots contributing to a particular hopeful candidate c are those ballots on which c is the topmost hopeful candidate. The quotient assigned to c is defined to be $q_c = v_c/(1+t_c)$, where v_c is the number of ballots contributing to c, and t_c is the sum of all the fractional numbers of candidates that those ballots have so far elected.

2.4. A ballot is *active* if it includes the name of a hopeful candidate (and is a valid ballot), and *inactive* otherwise. The *quota* is defined to be $v_{\mathrm{a}}/(1 + s - t_{\mathrm{x}})$, where v_{a} is the number of active ballots, s is the total number of seats to be filled, and t_{x} is the sum of the fractional numbers of candidates that are deemed to have been elected by all the *inactive* ballots.

2.5a. If c is the candidate with the highest quotient, and that quotient is greater than the quota, then c is declared elected. In this case each of the v_c ballots contributing to c is now deemed to have elected $1/q_c$ candidates *in total* (regardless of how many candidates it had elected before c's election); no change is made to the number of candidates elected by other ballots. (Since these v_c ballots collectively had previously elected t_c candidates, and they have now elected $v_c/q_c = 1 + t_c$ candidates, the sum of the fractional numbers of candidates elected by all voters has increased by 1.) If all s seats have now been filled, then the count ends; otherwise it proceeds to the next stage, from paragraph 2.3.

2.5b. If no candidate has a quotient greater than the quota, then the candidate with the smallest quotient is declared excluded. No change is made to the number of candidates elected by any ballot. If all but s candidates are now excluded, then all remaining hopeful candidates are declared elected and the count ends; otherwise the count proceeds to the next stage, from paragraph 2.3.

The details of the calculations of the quotients and quota may become clearer from a study of Election 2 in the next section.

The specification above contains two stopping conditions, in paragraphs 2.5a and 2.5b. These are included for convenience, to shorten the count. However, they are not necessary; they could be replaced by a single rule to the effect that the count ends when there are no hopeful candidates left. We shall see below (in Propositions 5 and 6) that, left to its own devices in this way, QPQ will elect exactly s candidates. It shares this property with Meek-STV but not with conventional STV, in which the stopping condition of paragaph 2.5b is needed in order to ensure that enough candidates are elected.

The most important proportionality property possessed by STV is what I call the *Droop proportionality criterion*: if more than k Droop quotas of voters are solidly committed to the same set of $l \geqslant k$ candidates, then at least k of those l candidates should be elected. (Here the *Droop quota* is the total number of valid ballots divided by one more than the number of seats to be filled, and a voter is *solidly committed* to a set of l candidates if the voter lists those candidates, in some order, as the top l candidates on their ballot.) We shall see in Proposition 7 that QPQ also satisfies the Droop proportionality criterion.

We shall see in Proposition 4 that if two candidates a and b are elected in successive stages, first a and then b, with no exclusion taking place between them, then b's quotient at the time of b's election is no greater than a's quotient at the time of a's election. (Thus with the d'Hondt–Phragmén method, which is essentially the same as QPQ but with no quota and no exclusions, each candidate elected has a quotient that is no greater than that of the previous candidate elected.)

This is not necessarily true, however, if an exclusion occurs between the elections of a and b. Consider the following election.

Election 1 (3 seats)

16 ab, 12 b, 12 c, 12 d, 8 eb.

There are 60 votes, and so the quota is $60/4 = 15$. The initial quotients are the numbers of first-preference votes; a, with a quotient of 16, exceeds the quota and is elected. Now b's quotient becomes $(16 + 12)/2 = 14$, and this is the only quotient to change, so that no other candidate reaches the quota. Thus e is excluded. Now b's quotient becomes $(16 + 12 + 8)/2 = 18$, and so b is elected with a quotient that is larger than a's was at the time of a's election. This means that each of the ab ballots was deemed to have elected $\frac{1}{16}$ of a candidate after a's election, but only $\frac{1}{18}$ of a candidate after b's election. This conveys the impression that these ballots have elected a negative proportion of b, or else (perhaps worse) that the b and eb ballots are being treated as having elected part of a.

To avoid this, it is proposed here that the count should be restarted from scratch after each exclusion. We shall see below, in Proposition 8, that if c is the first candidate to be excluded, and the count is then restarted with c's name deleted from all ballots, then all the candidates who were elected before c's exclusion will be elected again (although not necessarily first or in the same order). With this variant of the method, the count is divided into rounds, each of which apart from the last ends with an exclusion; the last round involves the election of s candidates in s successive stages, with no intervening exclusions. Now no ballot can ever be regarded as contributing a negative amount to any candidate, or a positive amount to a candidate not explicitly mentioned on it.

With Meek's method, a voter can tell from the result sheet exactly how their vote has been divided between the candidates mentioned on their ballot, and therefore how much they have contributed to the election of each candidate. QPQ does not explicitly divide votes between candidates; but with the multi-round version just described, as with the d'Hondt–Phragmén method itself, a voter can tell from the result sheet what proportion of each candidate they have elected; and multiplying these proportions by the final quota could be regarded as indicating how much of their vote has gone to each candidate, implicitly if not explicitly. For example, suppose candidates a and b are elected with quotients (at the time of election) $q_a > q_b$, candidate c is hopeful to the end, and the final quota is Q. Then a voter whose ballot (after the deletion of any excluded candidates) reads abc has elected $1/q_a$ of a, $1/q_b - 1/q_a$ of b, and was able to contribute $1/Q - 1/q_b$ towards the election of c (which, however, was insufficient to get c elected). And a voter whose ballot reads bac or bca has elected $1/q_b$ of b, nothing of a, and was again able to contribute $1/Q - 1/q_b$ towards the election of c. The fact that the abc and bac voters make the same contribution to c is a property that is shared with Meek-STV but not with conventional STV.

3 Examples

The first of these examples is intended to clarify the method of calculation of the quotients and quota.

Election 2 (3 seats)

5 a, 15 abc, 15 ac, 10 b, 15 bc,
20 c, 15 d, 5 e.

There are 100 votes, and so the initial quota is $100/4 = 25$. The initial quotients are the numbers of first-preference votes; a's quotient of 35 is the largest, and exceeds the quota, and so a is elected. Each of the 35 ballots that has a in first place is deemed to have elected $\frac{1}{35}$ of a; 5 of these plump for a and now become inactive, 15 have b in second place, and 15 have c in second place. So the quota now becomes $(100-5)/(4-\frac{5}{35}) \approx 24.62$, b's quotient becomes $(25+15)/(1+\frac{15}{35}) = 28.0$, and c's quotient becomes $(20+15)/(1+\frac{15}{35}) = 24.5$. Now b's quotient exceeds the quota, and so b is elected. Each of the 40 ballots that contributed to b's election is deemed to have elected $\frac{1}{28}$ of a candidate *in total*; 10 of these plump for b and now become inactive, and the remaining 30 have c in the place after b. So the quota now becomes $(100-5-10)/(4-\frac{5}{35}-\frac{10}{28}) \approx 24.29$, and c's quotient becomes $(20+15+30)/(1+\frac{15}{35}+\frac{30}{28}) = 26.0$. Now c is elected. We can set out the count as follows.

Election 2

	quotients					quota	result
	a	b	c	d	e		
Stage 1	35	25	20	15	5	25.00	a elected
Stage 2	–	28	$24\frac{1}{2}$	15	5	24.62	b elected
Stage 3	–	–	26	15	5	24.29	c elected

We have already mentioned that QPQ satisfies the Droop proportionality criterion, which is one important test of proportionality. The next two elections provide another test of proportionality. In both of these there are two parties, one with candidates a, b, c and the other with candidates d, e, f. The voters vote strictly along party lines. However, the abc-party voters all put a first, b second and c third, whereas the *def*-party voters are evenly divided among the three candidates. In Election 3, the abc party has just over half the votes, and so we expect it to gain 3 of the 5 seats, whereas in Election 4 it has just under half the votes, and so we expect it to gain only 2 seats. We shall see that this is what happens.

Election 3 (5 seats)		Election 4 (5 seats)	
306	abc	294	abc
99	def	103	def
98	efd	102	efd
97	fde	101	fde

In each case there are 600 votes, and so the quota is $600/6 = 100$. In Election 3, after the election of a, b and c the abc ballots become inactive, and, since these ballots are electing 3 seats, the quota reduces to $294/(6-3) = 98$. The counts proceed as follows.

Election 3

	quotients						quota	result
	a	b	c	d	e	f		
Stage 1	306	0	0	99	98	97	100	a elected
Stage 2	–	153	0	99	98	97	100	b elected
Stage 3	–	–	102	99	98	97	100	c elected
Stage 4	–	–	–	99	98	97	98	d elected
Stage 5	–	–	–	–	$98\frac{1}{2}$	97	98	e elected

Election 4

	quotients						quota	result
	a	b	c	d	e	f		
Stage 1	294	0	0	103	102	101	100	a elected
Stage 2	–	147	0	103	102	101	100	b elected
Stage 3	–	–	98	103	102	101	100	d elected
Stage 4	–	–	98	–	$102\frac{1}{2}$	101	100	e elected
Stage 5	–	–	98	–	–	102	100	f elected

We see that in each case the result is the one expected by proportionality. This is the same result as is obtained using STV (using the Droop quota—but not if the Hare quota is used).

In a single-seat election, QPQ and STV both reduce to the Alternative Vote. It is not clear how many seats and candidates are needed for QPQ to give a different result from Meek-STV, but here is an example with three seats and five candidates.

Election 5 (3 seats)

12 $acde$, 11 b, 7 cde, 8 dec, 9 ecd.

There are 47 votes, and so the quota (in STV or QPQ) is $47/4 = 11\frac{3}{4}$. STV elects a with a surplus of $\frac{1}{4}$ of a vote, which goes to c. No other candidate exceeds the quota, and so c, having the smallest vote, is excluded. Now d is elected with a surplus of $3\frac{1}{2}$ votes, which all goes to e, causing e to be elected. In QPQ, each candidate's initial quotient is their number of first-preference votes. So a is elected, and c's quotient then becomes $(12 + 7)/2 = 9\frac{1}{2}$. The candidate with the smallest quotient is now d, and so d is excluded. If the election is not restarted at this point, e now has a quotient of 17 and is elected, and this gives c a quotient of $(12 + 7 + 8 + 9)/3 = 12$ so that c is elected. If the election is restarted after d's exclusion, then e is elected first, and then there is a tie between a and c for the second place; whichever gets it, the other will get the third place. So in all cases the results are: STV: a, d, e; QPQ: a, c, e.

4 Proofs

In this section we will use the term *single-round QPQ* to refer to the version where one does not restart the count after an exclusion, and *multi-round QPQ* to refer to the version where one does. In the event that no exclusion occurs, both methods proceed identically, being then equivalent to the d'Hondt–Phragmén method. 'A count in which no exclusions occur' could refer to this possibility, in which exclusions are absent by chance, but it covers also the final round of a multi-round QPQ count, which is guaranteed to be free of exclusions; this final round is again equivalent to d'Hondt–Phragmén, although applied to ballots from which some candidates may already have been deleted.

It will be helpful to start by recalling some simple inequalities.

Proposition 1. *If m, n, x, y are positive real numbers such that $m/n \leqslant x/y$, then*

$$\frac{m}{n} \leqslant \frac{m+x}{n+y} \leqslant \frac{x}{y}\,. \tag{1.1}$$

If, in addition, $y < n$, then

$$\frac{m-x}{n-y} \leqslant \frac{m}{n}\,. \tag{1.2}$$

Proof. Since the denominators are all positive, the conclusions are equivalent to the inequalities $m(n+y) \leqslant (m+x)n$, $(m+x)y \leqslant x(n+y)$, and $(m-x)n \leqslant m(n-y)$. These all follow from the hypothesis, which is that $my \leqslant xn$. □

Proposition 2. *During a multi-round QPQ count, the quota never increases.*

Proof. To obtain a contradiction, suppose that the quota does increase at some stage, and consider the first stage at which this happens. Let the quota at the start of this stage be $Q = v_{\rm a}/(1 + s - t_{\rm x})$, where $v_{\rm a}$ is the number of active ballots at the start of this stage, and $t_{\rm x}$ is the sum of the fractional numbers of candidates that are deemed to have been elected by all the *inactive* ballots at the start of this stage. For each active ballot that becomes inactive in this stage, the effect is to subtract 1 from $v_{\rm a}$ and add t to $t_{\rm x}$, where t is the fractional number of candidates that that ballot has elected. This number t is either 0 or $1/q$, where q is the quotient possessed by some already-elected candidate at the time of their election. In order for this candidate to have been elected, necessarily q was greater than the quota at that time, which we are supposing was at least Q. Thus in all cases $t < 1/Q$. It follows that if x ballots become inactive in the current stage, then the effect is to subtract x from v_{a} and add a number $y < x/Q$ to t_{x}. Let Q' be the quota at the end of the current stage. If $y = 0$ then clearly $Q' < Q$. If $y \neq 0$ then $Q < x/y$, so that (1.2) gives

$$Q' = \frac{v_{\mathrm{a}} - x}{1 + s - t_{\mathrm{x}} - y} \leqslant \frac{v_{\mathrm{a}}}{1 + s - t_{\mathrm{x}}} = Q.$$

This contradicts the supposition that the quota increases in the current stage, and this contradiction proves the result. □

Proposition 3. *In any QPQ count, if a is elected with quotient q_a, and b is a hopeful candidate whose quotients at the start and end of the stage in which a is elected are q_b and q_b' respectively, then $q_b \leqslant q_b' \leqslant q_a$.*

Proof. Clearly $q_b \leqslant q_a$, since otherwise a would not have been elected in this stage. Suppose there are x ballots that contribute to a at the start of this stage and to b at the end of this stage, and let $y = x/q_a$, so that $x/y = q_a \geqslant q_b$. Then, after a's election, each of these x candidates is deemed to have elected $1/q_a$ candidates, so that collectively they have elected y candidates. If at the start of the current stage there were v_b ballots contributing to b, which collectively had already elected t_b candidates, then

$$q_b = \frac{v_b}{1 + t_b} \leqslant q_b' = \frac{v_b + x}{1 + t_b + y} \leqslant \frac{x}{y} = q_a$$

by (1.1). □

Proposition 4. *In a QPQ count in which no exclusions occur, each candidate to be elected has a quotient (at the time of election) that is no larger than the quotient (at the time of election) of the previous candidate to be elected.*

Proof. If candidates a and b are elected in successive stages, with quotients q_a and q_b' respectively, and if b's quotient at the start of the stage in which a is elected is q_b, then $q_b \leqslant q_b' \leqslant q_a$ by Proposition 3. In particular, $q_b' \leqslant q_a$, which is all we have to prove. □

Proposition 5. *Even if the stopping condition in paragraph 2.5a is deleted, it is not possible for more than s candidates to be elected by any form of QPQ (single-round or multi-round).*

Proof. Suppose it is. Consider the stage in which the $(s+1)$th candidate, c, is elected. At the start of this stage, let the quota be Q; let there be v_c ballots contributing to c, and suppose these v_c ballots collectively are currently electing t_c candidates; let there be v_{o} ballots contributing to other hopeful candidates, which are currently electing t_{o} candidates; let the number of active ballots be $v_{\mathrm{a}} = v_c + v_{\mathrm{o}}$; and let the number of candidates being elected by the inactive ballots be $t_{\mathrm{x}} = s - t_c - t_{\mathrm{o}}$. As in the proof of Proposition 2, every ballot has elected at most $1/Q$ candidates, and so $t_{\mathrm{o}} \leqslant v_{\mathrm{o}}/Q$. Thus

$$\frac{v_{\mathrm{o}}}{t_{\mathrm{o}}} \geqslant Q = \frac{v_{\mathrm{a}}}{1 + s - t_{\mathrm{x}}} = \frac{v_c + v_{\mathrm{o}}}{1 + t_c + t_{\mathrm{o}}},$$

and, by (1.2), c's quotient q_c satisfies

$$q_c = \frac{v_c}{1 + t_c} = \frac{(v_c + v_{\mathrm{o}}) - v_{\mathrm{o}}}{(1 + t_c + t_{\mathrm{o}}) - t_{\mathrm{o}}} \leqslant \frac{v_c + v_{\mathrm{o}}}{1 + t_c + t_{\mathrm{o}}} = Q.$$

This shows that c cannot be elected in the current stage, and this contradiction shows that at most s candidates are elected in total. □

Proposition 6. *Even if the stopping condition in paragraph 2.5b is deleted, at least s candidates must be elected by any form of QPQ (single-round or multi-round).*

Proof. Suppose this is not true, and consider the stage in which the number of nonexcluded candidates first falls below s. Suppose that at the start of this stage there are e elected candidates and (therefore) $s - e$ hopeful candidates. Since no hopeful candidate has a quotient greater than the quota,

$$v_c \leqslant Q(1 + t_c) \qquad (1.3)$$

for every hopeful candidate c, where Q is the quota, v_c is the number of ballots contributing to c, and t_c is the number of candidates that these ballots collectively have elected, all measured at the start of the current stage. Now, the sum of the $s - e$ numbers v_c is v_{a}, the number of active ballots, and the sum of the $s - e$ numbers t_c is the number of candidates elected by all the active ballots, which is $e - t_{\mathrm{x}}$, where t_{x} is the number of candidates elected by the inactive ballots. So summing (1.3) over all $s-e$ hopeful candidates gives $v_{\mathrm{a}} \leqslant Q(s-e+e-t_{\mathrm{x}}) = Q(s-t_{\mathrm{x}})$. Thus

$$Q = \frac{v_{\mathrm{a}}}{1+s-t_{\mathrm{x}}} < \frac{v_{\mathrm{a}}}{s-t_{\mathrm{x}}} \leqslant Q.$$

This contradiction shows that at least one of the $s-e$ hopeful candidates must have a quotient greater than the quota Q, and so be elected in the current stage. This contradicts the supposition that the number of nonexcluded candidates falls in the current stage, and this contradiction proves the result. □

Propositions 5 and 6 together show that, left to its own devices, QPQ will always elect the right number of candidates; the only stopping condition required is that the election must terminate when there are no hopeful candidates left.

Proposition 7. *Every form of QPQ satisfies the Droop proportionality criterion: if more than k Droop quotas of voters are solidly committed to the same set of $l \geqslant k$ candidates, then at least k of those l candidates must be elected.*

Proof. The argument is rather similar to the proof of the previous proposition. Let L be the set of l candidates in question. In view of Proposition 5, we may assume that the stopping condition in paragraph 2.5a is deleted, so that the count cannot end because we have elected too many candidates outside L. Thus if Proposition 7 is not true then there must come a point in some election at which the number of nonexcluded candidates in L falls below k. Consider the stage in which this happens. Suppose that at the start of this stage there are e elected candidates and (therefore) $k-e$ hopeful candidates in L. Since no hopeful candidate has a quotient greater than the quota Q, (1.3) holds for all these $k-e$ hopeful candidates. Since the quota at the start of the count was equal to the Droop quota, and, by Proposition 2, the quota never increases, the number of ballots solidly committed to L is greater than kQ, and so the sum of the $k-e$ numbers v_c is greater than kQ. Moreover, none of these ballots can have contributed to electing any candidate outside L, and so the sum of the $k-e$ numbers t_c is at most e. So summing (1.3) over all $k-e$ hopeful candidates in L gives

$$Qk < \sum_c v_c \leqslant Q\Big(\sum_c (1+t_c)\Big) \leqslant Q(k-e+e) = Qk.$$

This contradiction shows that at least one of the hopeful candidates in L must have a quotient that is greater than Q, and so the number of nonexcluded candidates in L cannot fall in the current stage. This contradiction in turn proves the result. □

Proposition 8. *Suppose that in the first k stages of a QPQ count candidates $a_1, \ldots, a_k$ are elected (in that order) with quotients $q_1, \ldots, q_k$ respectively, and in the $(k+1)$th stage candidate b is excluded. Suppose that the count is restarted with b's name deleted from every ballot. Then, in the new count, candidates $a_1, \ldots, a_k$ will all be elected before any exclusions take place, and each candidate a_i will have quotient at least q_i at the time of their election.*

Proof. Suppose that in the first count the quota at the time of a_i's election is Q_i, so that $q_i > Q_i$, for each i. The deletion of b cannot decrease any candidate's initial quotient, nor increase the quota, and so at the start of the new count a_1 has quotient at least q_1 and the quota is at most Q_1. Since, by Propositions 2 and 3, the election of other candidates cannot increase the quota nor decrease a_1's quotient, a_1 will have a quotient greater than the quota as long as a_1 remains hopeful. Thus a_1 will eventually be elected, before any exclusions take place, with a quotient that is at least q_1.

In order to obtain a contradiction, suppose that the conclusion of the Proposition does not hold for all these values of i, and consider the smallest value of i for which it fails to hold. Then $i \geqslant 2$, since we have just seen that the conclusion holds for a_1. Consider the first point at which $a_1, a_2, \ldots, a_{i-1}$ are all elected, and let a_j be the last of these candidates to be elected; a_j may, but need not, be a_{i-1}. Since the conclusion holds for all of $a_1, a_2, \ldots, a_{i-1}$, we know that a_j had quotient at least q_j at the time of election. By Proposition 4 applied to the first count and then to the new count, $q_j \geqslant q_i$, and every candidate elected so far in the new count has been elected with a quotient that is at least q_j and hence at least q_i. So if a_i has already been elected in the new count then the conclusion of the Proposition holds for a_i. Since we are supposing that this is not the case, it must be that a_i has not yet been elected. We will consider a_i's quotient and the quota at the start of the next stage, immediately following the election of a_j.

In the first count, a_i was elected with quotient $q_i = v_i/(1+t_i)$, where v_i is the number of ballots that contributed to a_i after a_{i-1}'s election, and t_i is the fractional number of candidates that these ballots had so far elected. These v_i ballots are the ones on which no candidate other than $a_1, \ldots, a_{i-1}$ is preferred to a_i, and so they again contribute to a_i at this point in the new count. So in the new count, a_i now has quotient $\hat{q}_i = (v_i + v'_i)/(1 + \hat{t}_i + \hat{t}'_i)$, where v'_i is the number of ballots contributing to a_i at this point that did not contribute to a_i at the time of a_i's election in the first count, and $\hat{t}_i$ and $\hat{t}'_i$ are the fractional numbers of candidates elected by the original v_i contributors and the new v'_i contributors at this point in the new count. Each of these $v_i + v'_i$ ballots is deemed to have elected either 0 candidates or a number of candidates of the form $1/\hat{q}$, where $\hat{q}$ is the smallest quotient of any elected candidate listed above a_i on that ballot. For all the ballots of this second type, $\hat{q} \geqslant q_j \geqslant q_i$; thus $\hat{t}'_i \leqslant v'_i/q_i$ and $v'_i/\hat{t}'_i \geqslant q_i$. Moreover, for each of the original v_i ballots that is of this second type, the number $\hat{q}$ for that ballot is the smallest of a new set of quotients, each of which is at least as large as the corresponding quotient in the original count, so that if the ballot was electing $1/q$ candidates at the time of a_i's election in the original count then $\hat{q} \geqslant q$ and $1/\hat{q} \leqslant 1/q$; thus $\hat{t}_i \leqslant t_i$. It follows from (1.1) that

$$\hat{q}_i = \frac{v_i + v'_i}{1 + \hat{t}_i + \hat{t}'_i} \geqslant \frac{v_i + v'_i}{1 + t_i + \hat{t}'_i} \geqslant \frac{v_i}{1 + t_i} = q_i. \quad (1.4)$$

Now let us consider the quota. Let v be the number of valid ballots. In the first count, the quota at the time of a_i's election was $Q_i = (v - v_x)/(1 + s - t_x)$, where v_x is the number of *inactive* ballots at the time of a_i's election, and t_x is the fractional number of candidates that these ballots have elected. These v_x inactive ballots are the ones that contain the name of no candidates other than $a_1, \ldots, a_{i-1}$, and so they are again inactive at this point in the new count. So in the new count, the quota at this point is $\hat{Q}_i = (v - v_x - v'_x)/(1 + s - \hat{t}_x - \hat{t}'_x)$, where v'_x is the number of ballots that were active at the time of a_i's election in the first count but are inactive at this point in the new count, and $\hat{t}_x$ and $\hat{t}'_x$ are the fractional numbers of candidates elected by the original and the new inactive ballots at this point in the new count. By the same argument we used in the previous paragraph to prove that $\hat{t}_i \leqslant t_i$, we can now deduce that $\hat{t}_x \leqslant t_x$. Moreover, by Propositions 2 and 4 and the criterion for election in paragraph 2.5a, every candidate elected so far has been elected with a quotient that is greater than the current quota $\hat{Q}_i$, so that $\hat{t}'_x \leqslant v'_x/\hat{Q}_i$ and $v'_x/t'_x \geqslant \hat{Q}_i$. It follows from (1.1) that

$$\hat{Q}_i = \frac{v - v_x - v'_x}{1 + s - \hat{t}_x - \hat{t}'_x} \leqslant \frac{v - v_x}{1 + s - \hat{t}_x} \leqslant \frac{v - v_x}{1 + s - t_x} = Q_i. \quad (1.5)$$

It follows from (1.4) and (1.5) that $\hat{q}_i \geqslant q_i > Q_i \geqslant \hat{Q}_i$, so that a_i's current quotient is greater than the current quota. Since, by Propositions 2 and 3, the election of other candidates cannot increase the quota nor decrease a_i's quotient, a_i will have a quotient greater than the quota as long as a_i remains hopeful. Thus a_i will eventually be elected, before any exclusions take place, with a quotient that is at least q_i. This contradicts the supposition that the conclusion of the Proposition failed to hold for a_i, and this contradiction completes the proof of the Proposition. □

5 References

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