Voting matters

for the technical issues of STV

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Editorial

This is the first issue under the auspices of the McDougall Trust. The Editor has taken the opportunity of this change to make a number of stylistic changes. These are mainly as follows:

- Use of the LaTeX typesetting system so that, if they wish, authors can submit material in a format that can be directly typeset.
- Starting papers on a new page so that individual papers can be handled more easily.

This issue also has a slight departure in having two papers which are more mathematical in nature than is usual. It has been decided that the Editor should ensure that the main points of such papers are intelligible to non-mathematical readers by placing an appropriate summary here.

There are four papers in this issue:

- D R Woodall: *QPQ, a quota-preferential STV-like election rule*,
- J Otten: *Fuller Disclosure than Intended*,
- M Schulze: *A New Monotonic and Clone-Independent Single-Winner Election Method* and
- J Gilmour: *Calculation of Transfer Values — Proposal for STV-PR Rules for Local Government Elections in Scotland*.

In Douglas Woodall's paper he defines a new way of counting preferential votes which is analogous to conventional STV. To understand the counting process, it is probably best to work through the examples in the paper with the general definition in mind. It is clear that undertaking this form of counting without a computer is viable. Hence the interest here would be to see if QPQ has any appeal to those who think it inappropriate to use computers to count an election. The main mathematics in Woodall's paper is to show that QPQ has several desirable properties — hence this part can be skipped and the results taken on trust.

The paper of Joe Otten arose from a resolution put to the ERS AGM requesting that the full election data of the preferences specified should be available for STV elections. (Such disclosure was available for the three Irish constituencies for which electronic voting was employed in the June 2002 elections.) The paper explains a potential danger from full disclosure with a proposed resolution.

Markus Schulze in his paper considers the question of electing just one person, which would be the Alternative Vote (AV) with STV. Many would consider that AV is inappropriate since it does not necessarily elect the Condorcet winner (if there is one). The paper starts from the position of electing the Condorcet winner but with the objective of ensuring as many desirable properties are satisfied as possible. The proof that certain properties are satisfied involves some logical analysis which I hope most readers can follow.

James Gilmour's paper has arisen as a result of the recent consultation process for the introduction of STV in Scottish local elections. Here, he shows by analysis and example that the calculation of the transfer values can be improved by using more precision in the calculation than is often the case.

QPQ, a quota-preferential STV-like election rule

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1 Introduction

Olli Salmi, in a posting to an Election Methods list [6], has suggested a new quota-preferential election rule, which is developed slightly further in this article, and which is remarkably similar to the Single Transferable Vote (STV) in its effects. I shall call it QPQ, for Quota-Preferential by Quotient. Both in its properties and in the results it gives, it seems to be more like Meek's version of STV [2] than the traditional version [3]. This is surprising since: (i) in marked contrast with STV, the quota in QPQ is used only as a criterion for election, and not in the transfer of surplus votes; (ii) QPQ, unlike Meek's method, involves no iterative processes, and so the votes can be counted by hand; and (iii) QPQ derives from the European continental tradition of party list systems (specifically, d'Hondt's rule), which is usually regarded as quite different from STV. I do not imagine that anyone who is already using STV will see any reason to switch to QPQ; but people who are already using d'Hondt's rule may feel that QPQ is a natural progression of it, and so more acceptable than STV.

D'Hondt's rule for allocating seats to parties was proposed by the Belgian lawyer Victor d'Hondt [1] in 1882. The seats are allocated to the parties one by one. At each stage, a party with v votes and (currently) s seats is assigned the quotient $v/(1+s)$, and the next seat is allocated to the party with the largest quotient. This continues until all seats have been filled.

Many variations of this rule were subsequently proposed, in which the divisor $1+s$ is replaced by some other function of s. However, the next contribution of relevance to us is an adaptation of d'Hondt's rule to work with STV-type preferential ballots. This adaptation has been part of Sweden's Elections Act for many years; we will call it the *d'Hondt–Phragmén method*, since it is based on a method proposed by the Swedish mathematician Lars Edvard Phragmén [4, 5] in 1895. The seats are again allocated one by one, only this time to candidates rather than parties; at each stage, the next seat is allocated to the candidate with the largest quotient (calculated as explained below). In the event that the voters effectively vote for disjoint party lists (e.g., if every ballot is marked for *abcd*, *efg* or *hijkl*), then the d'Hondt–Phragmén method gives exactly the same result as d'Hondt's rule. However, it was introduced in the Swedish Elections Act as a means of allocating seats *within* a party, at a time when voters were allowed to express a choice of candidates within the party. It does not guarantee to represent minorities proportionally.

Salmi's contribution has been to introduce a quota into Phragmén's method. In this version, which he calls the *d'Hondt–Phragmén method with quota*, the candidate with the largest quotient will get the next seat if, and only if, this quotient is larger than the quota; otherwise, the candidate with the smallest quotient is excluded, and the quotients are recalculated. In this respect it is like STV. However, unlike in STV, this is the only way in which the quota is used; it is not used in transferring votes. QPQ, as described here, differs from Salmi's original version only in that the quota is defined slightly differently, and the count is preferably restarted after every exclusion.

Both the d'Hondt–Phragmén method (with or without quota), and QPQ, can be described in terms of groups of voters rather than individuals, and this is naturally how one thinks when processing piles of ballots by hand. But it seems to me that they are easier to understand when rewritten in terms of individual ballots rather than groups, and they are described here in this form. From now on, s denotes the total number of seats to be filled.

2 The details of QPQ

2.1. The count is divided into a sequence of stages. At the start of each stage, each candidate is in one of three states, designated as *elected*, *excluded* and *hopeful*.

At the start of the first stage, every candidate is hopeful. In each stage, either one hopeful candidate is reclassified as elected, or one hopeful candidate is reclassified as excluded.

2.2. At the start of each stage, each ballot is deemed to have elected some fractional number of candidates, in such a way that the sum of these fractional numbers over all ballots is equal to the number of candidates who are currently classed as elected. At the start of the first stage, every ballot has elected 0 candidates.

2.3. At the start of each stage, the quotients of all the hopeful candidates are calculated, as follows. The ballots contributing to a particular hopeful candidate c are those ballots on which c is the topmost hopeful candidate. The quotient assigned to c is defined to be $q_c = v_c/(1+t_c)$, where v_c is the number of ballots contributing to c, and t_c is the sum of all the fractional numbers of candidates that those ballots have so far elected.

2.4. A ballot is *active* if it includes the name of a hopeful candidate (and is a valid ballot), and *inactive* otherwise. The *quota* is defined to be $v_{\mathrm{a}}/(1 + s - t_{\mathrm{x}})$, where v_{a} is the number of active ballots, s is the total number of seats to be filled, and t_{x} is the sum of the fractional numbers of candidates that are deemed to have been elected by all the *inactive* ballots.

2.5a. If c is the candidate with the highest quotient, and that quotient is greater than the quota, then c is declared elected. In this case each of the v_c ballots contributing to c is now deemed to have elected $1/q_c$ candidates *in total* (regardless of how many candidates it had elected before c's election); no change is made to the number of candidates elected by other ballots. (Since these v_c ballots collectively had previously elected t_c candidates, and they have now elected $v_c/q_c = 1 + t_c$ candidates, the sum of the fractional numbers of candidates elected by all voters has increased by 1.) If all s seats have now been filled, then the count ends; otherwise it proceeds to the next stage, from paragraph 2.3.

2.5b. If no candidate has a quotient greater than the quota, then the candidate with the smallest quotient is declared excluded. No change is made to the number of candidates elected by any ballot. If all but s candidates are now excluded, then all remaining hopeful candidates are declared elected and the count ends; otherwise the count proceeds to the next stage, from paragraph 2.3.

The details of the calculations of the quotients and quota may become clearer from a study of Election 2 in the next section.

The specification above contains two stopping conditions, in paragraphs 2.5a and 2.5b. These are included for convenience, to shorten the count. However, they are not necessary; they could be replaced by a single rule to the effect that the count ends when there are no hopeful candidates left. We shall see below (in Propositions 5 and 6) that, left to its own devices in this way, QPQ will elect exactly s candidates. It shares this property with Meek-STV but not with conventional STV, in which the stopping condition of paragaph 2.5b is needed in order to ensure that enough candidates are elected.

The most important proportionality property possessed by STV is what I call the *Droop proportionality criterion*: if more than k Droop quotas of voters are solidly committed to the same set of $l \geqslant k$ candidates, then at least k of those l candidates should be elected. (Here the *Droop quota* is the total number of valid ballots divided by one more than the number of seats to be filled, and a voter is *solidly committed* to a set of l candidates if the voter lists those candidates, in some order, as the top l candidates on their ballot.) We shall see in Proposition 7 that QPQ also satisfies the Droop proportionality criterion.

We shall see in Proposition 4 that if two candidates a and b are elected in successive stages, first a and then b, with no exclusion taking place between them, then b's quotient at the time of b's election is no greater than a's quotient at the time of a's election. (Thus with the d'Hondt–Phragmén method, which is essentially the same as QPQ but with no quota and no exclusions, each candidate elected has a quotient that is no greater than that of the previous candidate elected.)

This is not necessarily true, however, if an exclusion occurs between the elections of a and b. Consider the following election.

Election 1 (3 seats)

16 ab, 12 b, 12 c, 12 d, 8 eb.

There are 60 votes, and so the quota is $60/4 = 15$. The initial quotients are the numbers of first-preference votes; a, with a quotient of 16, exceeds the quota and is elected. Now b's quotient becomes $(16 + 12)/2 = 14$, and this is the only quotient to change, so that no other candidate reaches the quota. Thus e is excluded. Now b's quotient becomes $(16+12+8)/2 = 18$, and so b is elected with a quotient that is larger than a's was at the time of a's election. This means that each of the ab ballots was deemed to have elected $\frac{1}{16}$ of a candidate after a's election, but only $\frac{1}{18}$ of a candidate after b's election. This conveys the impression that these ballots have elected a negative proportion of b, or else (perhaps worse) that the b and eb ballots are being treated as having elected part of a.

To avoid this, it is proposed here that the count should be restarted from scratch after each exclusion. We shall see below, in Proposition 8, that if c is the first candidate to be excluded, and the count is then restarted with c's name deleted from all ballots, then all the candidates who were elected before c's exclusion will be elected again (although not necessarily first or in the same order). With this variant of the method, the count is divided into rounds, each of which apart from the last ends with an exclusion; the last round involves the election of s candidates in s successive stages, with no intervening exclusions. Now no ballot can ever be regarded as contributing a negative amount to any candidate, or a positive amount to a candidate not explicitly mentioned on it.

With Meek's method, a voter can tell from the result sheet exactly how their vote has been divided between the candidates mentioned on their ballot, and therefore how much they have contributed to the election of each candidate. QPQ does not explicitly divide votes between candidates; but with the multi-round version just described, as with the d'Hondt–Phragmén method itself, a voter can tell from the result sheet what proportion of each candidate they have elected; and multiplying these proportions by the final quota could be regarded as indicating how much of their vote has gone to each candidate, implicitly if not explicitly. For example, suppose candidates a and b are elected with quotients (at the time of election) $q_a > q_b$, candidate c is hopeful to the end, and the final quota is Q. Then a voter whose ballot (after the deletion of any excluded candidates) reads abc has elected $1/q_a$ of a, $1/q_b - 1/q_a$ of b, and was able to contribute $1/Q - 1/q_b$ towards the election of c (which, however, was insufficient to get c elected). And a voter whose ballot reads bac or bca has elected $1/q_b$ of b, nothing of a, and was again able to contribute $1/Q - 1/q_b$ towards the election of c. The fact that the abc and bac voters make the same contribution to c is a property that is shared with Meek-STV but not with conventional STV.

3 Examples

The first of these examples is intended to clarify the method of calculation of the quotients and quota.

Election 2 (3 seats)

5 a, 15 abc, 15 ac, 10 b, 15 bc,
20 c, 15 d, 5 e.

There are 100 votes, and so the initial quota is $100/4 = 25$. The initial quotients are the numbers of first-preference votes; a's quotient of 35 is the largest, and exceeds the quota, and so a is elected. Each of the 35 ballots that has a in first place is deemed to have elected $\frac{1}{35}$ of a; 5 of these plump for a and now become inactive, 15 have b in second place, and 15 have c in second place. So the quota now becomes $(100-5)/(4-\frac{5}{35}) \approx 24.62$, b's quotient becomes $(25+15)/(1+\frac{15}{35}) = 28.0$, and c's quotient becomes $(20+15)/(1+\frac{15}{35}) = 24.5$. Now b's quotient exceeds the quota, and so b is elected. Each of the 40 ballots that contributed to b's election is deemed to have elected $\frac{1}{28}$ of a candidate *in total*; 10 of these plump for b and now become inactive, and the remaining 30 have c in the place after b. So the quota now becomes $(100-5-10)/(4-\frac{5}{35}-\frac{10}{28}) \approx 24.29$, and c's quotient becomes $(20+15+30)/(1+\frac{15}{35}+\frac{30}{28}) = 26.0$. Now c is elected. We can set out the count as follows.

Election 2

	quotients					quota	result
	a	b	c	d	e		
Stage 1	35	25	20	15	5	25.00	a elected
Stage 2	–	28	$24\frac{1}{2}$	15	5	24.62	b elected
Stage 3	–	–	26	15	5	24.29	c elected

We have already mentioned that QPQ satisfies the Droop proportionality criterion, which is one important test of proportionality. The next two elections provide another test of proportionality. In both of these there are two parties, one with candidates a, b, c and the other with candidates d, e, f. The voters vote strictly along party lines. However, the abc-party voters all put a first, b second and c third, whereas the def-party voters are evenly divided among the three candidates. In Election 3, the abc party has just over half the votes, and so we expect it to gain 3 of the 5 seats, whereas in Election 4 it has just under half the votes, and so we expect it to gain only 2 seats. We shall see that this is what happens.

Election 3 (5 seats)		Election 4 (5 seats)	
306	abc	294	abc
99	def	103	def
98	efd	102	efd
97	fde	101	fde

In each case there are 600 votes, and so the quota is $600/6 = 100$. In Election 3, after the election of a, b and c the abc ballots become inactive, and, since these ballots are electing 3 seats, the quota reduces to $294/(6-3) = 98$. The counts proceed as follows.

Election 3

	quotients						quota	result
	a	b	c	d	e	f		
Stage 1	306	0	0	99	98	97	100	a elected
Stage 2	–	153	0	99	98	97	100	b elected
Stage 3	–	–	102	99	98	97	100	c elected
Stage 4	–	–	–	99	98	97	98	d elected
Stage 5	–	–	–	–	$98\frac{1}{2}$	97	98	e elected

Election 4

	quotients						quota	result
	a	b	c	d	e	f		
Stage 1	294	0	0	103	102	101	100	a elected
Stage 2	–	147	0	103	102	101	100	b elected
Stage 3	–	–	98	103	102	101	100	d elected
Stage 4	–	–	98	–	$102\frac{1}{2}$	101	100	e elected
Stage 5	–	–	98	–	–	102	100	f elected

We see that in each case the result is the one expected by proportionality. This is the same result as is obtained using STV (using the Droop quota—but not if the Hare quota is used).

In a single-seat election, QPQ and STV both reduce to the Alternative Vote. It is not clear how many seats and candidates are needed for QPQ to give a different result from Meek-STV, but here is an example with three seats and five candidates.

Election 5 (3 seats)

12 $acde$, 11 b, 7 cde, 8 dec, 9 ecd.

There are 47 votes, and so the quota (in STV or QPQ) is $47/4 = 11\frac{3}{4}$. STV elects a with a surplus of $\frac{1}{4}$ of a vote, which goes to c. No other candidate exceeds the quota, and so c, having the smallest vote, is excluded. Now d is elected with a surplus of $3\frac{1}{2}$ votes, which all goes to e, causing e to be elected. In QPQ, each candidate's initial quotient is their number of first-preference votes. So a is elected, and c's quotient then becomes $(12 + 7)/2 = 9\frac{1}{2}$. The candidate with the smallest quotient is now d, and so d is excluded. If the election is not restarted at this point, e now has a quotient of 17 and is elected, and this gives c a quotient of $(12 + 7 + 8 + 9)/3 = 12$ so that c is elected. If the election is restarted after d's exclusion, then e is elected first, and then there is a tie between a and c for the second place; whichever gets it, the other will get the third place. So in all cases the results are: STV: a, d, e; QPQ: a, c, e.

4 Proofs

In this section we will use the term *single-round QPQ* to refer to the version where one does not restart the count after an exclusion, and *multi-round QPQ* to refer to the version where one does. In the event that no exclusion occurs, both methods proceed identically, being then equivalent to the d'Hondt–Phragmén method. 'A count in which no exclusions occur' could refer to this possibility, in which exclusions are absent by chance, but it covers also the final round of a multi-round QPQ count, which is guaranteed to be free of exclusions; this final round is again equivalent to d'Hondt–Phragmén, although applied to ballots from which some candidates may already have been deleted.

It will be helpful to start by recalling some simple inequalities.

Proposition 1. *If m, n, x, y are positive real numbers such that $m/n \leqslant x/y$, then*

$$\frac{m}{n} \leqslant \frac{m+x}{n+y} \leqslant \frac{x}{y}\,. \tag{1.1}$$

If, in addition, $y < n$, then

$$\frac{m-x}{n-y} \leqslant \frac{m}{n}\,. \tag{1.2}$$

Proof. Since the denominators are all positive, the conclusions are equivalent to the inequalities $m(n+y) \leqslant (m+x)n$, $(m+x)y \leqslant x(n+y)$, and $(m-x)n \leqslant m(n-y)$. These all follow from the hypothesis, which is that $my \leqslant xn$. $\square$

Proposition 2. *During a multi-round QPQ count, the quota never increases.*

Proof. To obtain a contradiction, suppose that the quota does increase at some stage, and consider the first stage at which this happens. Let the quota at the start of this stage be $Q = v_{\mathrm{a}}/(1 + s - t_{\mathrm{x}})$, where v_{a} is the number of active ballots at the start of this stage, and t_{x} is the sum of the fractional numbers of candidates that are deemed to have been elected by all the *inactive* ballots at the start of this stage. For each active ballot that becomes inactive in this stage, the effect is to subtract 1 from v_{a} and add t to t_{x}, where t is the fractional number of candidates that that ballot has elected. This number t is either 0 or $1/q$, where q is the quotient possessed by some already-elected candidate at the time of their election. In order for this candidate to have been elected, necessarily q was greater than the quota at that time, which we are supposing was at least Q. Thus in all cases $t < 1/Q$. It follows that if x ballots become inactive in the current stage, then the effect is to subtract x from v_{a} and add a number $y < x/Q$ to t_{x}. Let Q' be the quota at the end of the current stage. If $y = 0$ then clearly $Q' < Q$. If $y \neq 0$ then $Q < x/y$, so that (1.2) gives

$$Q' = \frac{v_{\mathrm{a}} - x}{1 + s - t_{\mathrm{x}} - y} \leqslant \frac{v_{\mathrm{a}}}{1 + s - t_{\mathrm{x}}} = Q.$$

This contradicts the supposition that the quota increases in the current stage, and this contradiction proves the result. □

Proposition 3. *In any QPQ count*, *if a is elected with quotient q_a, and b is a hopeful candidate whose quotients at the start and end of the stage in which a is elected are q_b and q'_b respectively, then $q_b \leqslant q'_b \leqslant q_a$.*

Proof. Clearly $q_b \leqslant q_a$, since otherwise a would not have been elected in this stage. Suppose there are x ballots that contribute to a at the start of this stage and to b at the end of this stage, and let $y = x/q_a$, so that $x/y = q_a \geqslant q_b$. Then, after a's election, each of these x candidates is deemed to have elected $1/q_a$ candidates, so that collectively they have elected y candidates. If at the start of the current stage there were v_b ballots contributing to b, which collectively had already elected t_b candidates, then

$$q_b = \frac{v_b}{1 + t_b} \leqslant q'_b = \frac{v_b + x}{1 + t_b + y} \leqslant \frac{x}{y} = q_a$$

by (1.1). □

Proposition 4. *In a QPQ count in which no exclusions occur, each candidate to be elected has a quotient* (*at the time of election*) *that is no larger than the quotient* (*at the time of election*) *of the previous candidate to be elected.*

Proof. If candidates a and b are elected in successive stages, with quotients q_a and q'_b respectively, and if b's quotient at the start of the stage in which a is elected is q_b, then $q_b \leqslant q'_b \leqslant q_a$ by Proposition 3. In particular, $q'_b \leqslant q_a$, which is all we have to prove. □

Proposition 5. *Even if the stopping condition in paragraph 2.5a is deleted*, *it is not possible for more than s candidates to be elected by any form of QPQ* (*single-round or multi-round*).

Proof. Suppose it is. Consider the stage in which the $(s+1)$th candidate, c, is elected. At the start of this stage, let the quota be Q; let there be v_c ballots contributing to c, and suppose these v_c ballots collectively are currently electing t_c candidates; let there be v_{o} ballots contributing to other hopeful candidates, which are currently electing t_{o} candidates; let the number of active ballots be $v_{\mathrm{a}} = v_c + v_{\mathrm{o}}$; and let the number of candidates being elected by the inactive ballots be $t_{\mathrm{x}} = s - t_c - t_{\mathrm{o}}$. As in the proof of Proposition 2, every ballot has elected at most $1/Q$ candidates, and so $t_{\mathrm{o}} \leqslant v_{\mathrm{o}}/Q$. Thus

$$\frac{v_{\mathrm{o}}}{t_{\mathrm{o}}} \geqslant Q = \frac{v_{\mathrm{a}}}{1 + s - t_{\mathrm{x}}} = \frac{v_c + v_{\mathrm{o}}}{1 + t_c + t_{\mathrm{o}}},$$

and, by (1.2), c's quotient q_c satisfies

$$q_c = \frac{v_c}{1 + t_c} = \frac{(v_c + v_{\mathrm{o}}) - v_{\mathrm{o}}}{(1 + t_c + t_{\mathrm{o}}) - t_{\mathrm{o}}} \leqslant \frac{v_c + v_{\mathrm{o}}}{1 + t_c + t_{\mathrm{o}}} = Q.$$

This shows that c cannot be elected in the current stage, and this contradiction shows that at most s candidates are elected in total. □

Proposition 6. *Even if the stopping condition in paragraph 2.5b is deleted*, *at least s candidates must be elected by any form of QPQ* (*single-round or multi-round*).

Proof. Suppose this is not true, and consider the stage in which the number of nonexcluded candidates first falls below s. Suppose that at the start of this stage there are e elected candidates and (therefore) $s - e$ hopeful candidates. Since no hopeful candidate has a quotient greater than the quota,

$$v_c \leqslant Q(1 + t_c) \tag{1.3}$$

for every hopeful candidate c, where Q is the quota, v_c is the number of ballots contributing to c, and t_c is the number of candidates that these ballots collectively have elected, all measured at the start of the current stage. Now, the sum of the $s - e$ numbers v_c is v_{a}, the number of active ballots, and the sum of the $s - e$ numbers t_c is the number of candidates elected by all the active ballots, which is $e - t_{\mathrm{x}}$, where t_{x} is the number of candidates elected by the inactive ballots. So summing (1.3) over all $s-e$ hopeful candidates gives $v_{\mathrm{a}} \leqslant Q(s-e+e-t_{\mathrm{x}}) = Q(s-t_{\mathrm{x}})$. Thus

$$Q = \frac{v_{\mathrm{a}}}{1+s-t_{\mathrm{x}}} < \frac{v_{\mathrm{a}}}{s-t_{\mathrm{x}}} \leqslant Q.$$

This contradiction shows that at least one of the $s-e$ hopeful candidates must have a quotient greater than the quota Q, and so be elected in the current stage. This contradicts the supposition that the number of nonexcluded candidates falls in the current stage, and this contradiction proves the result. □

Propositions 5 and 6 together show that, left to its own devices, QPQ will always elect the right number of candidates; the only stopping condition required is that the election must terminate when there are no hopeful candidates left.

Proposition 7. *Every form of QPQ satisfies the Droop proportionality criterion: if more than k Droop quotas of voters are solidly committed to the same set of $l \geqslant k$ candidates, then at least k of those l candidates must be elected.*

Proof. The argument is rather similar to the proof of the previous proposition. Let L be the set of l candidates in question. In view of Proposition 5, we may assume that the stopping condition in paragraph 2.5a is deleted, so that the count cannot end because we have elected too many candidates outside L. Thus if Proposition 7 is not true then there must come a point in some election at which the number of nonexcluded candidates in L falls below k. Consider the stage in which this happens. Suppose that at the start of this stage there are e elected candidates and (therefore) $k-e$ hopeful candidates in L. Since no hopeful candidate has a quotient greater than the quota Q, (1.3) holds for all these $k-e$ hopeful candidates. Since the quota at the start of the count was equal to the Droop quota, and, by Proposition 2, the quota never increases, the number of ballots solidly committed to L is greater than kQ, and so the sum of the $k-e$ numbers v_c is greater than kQ. Moreover, none of these ballots can have contributed to electing any candidate outside L, and so the sum of the $k-e$ numbers t_c is at most e. So summing (1.3) over all $k-e$ hopeful candidates in L gives

$$Qk < \sum_c v_c \leqslant Q\Big(\sum_c (1+t_c)\Big) \leqslant Q(k-e+e) = Qk.$$

This contradiction shows that at least one of the hopeful candidates in L must have a quotient that is greater than Q, and so the number of nonexcluded candidates in L cannot fall in the current stage. This contradiction in turn proves the result. □

Proposition 8. *Suppose that in the first k stages of a QPQ count candidates $a_1, \ldots, a_k$ are elected (in that order) with quotients $q_1, \ldots, q_k$ respectively, and in the $(k+1)$th stage candidate b is excluded. Suppose that the count is restarted with b's name deleted from every ballot. Then, in the new count, candidates $a_1, \ldots, a_k$ will all be elected before any exclusions take place, and each candidate a_i will have quotient at least q_i at the time of their election.*

Proof. Suppose that in the first count the quota at the time of a_i's election is Q_i, so that $q_i > Q_i$, for each i. The deletion of b cannot decrease any candidate's initial quotient, nor increase the quota, and so at the start of the new count a_1 has quotient at least q_1 and the quota is at most Q_1. Since, by Propositions 2 and 3, the election of other candidates cannot increase the quota nor decrease a_1's quotient, a_1 will have a quotient greater than the quota as long as a_1 remains hopeful. Thus a_1 will eventually be elected, before any exclusions take place, with a quotient that is at least q_1.

In order to obtain a contradiction, suppose that the conclusion of the Proposition does not hold for all these values of i, and consider the smallest value of i for which it fails to hold. Then $i \geqslant 2$, since we have just seen that the conclusion holds for a_1. Consider the first point at which $a_1, a_2, \ldots, a_{i-1}$ are all elected, and let a_j be the last of these candidates to be elected; a_j may, but need not, be a_{i-1}. Since the conclusion holds for all of $a_1, a_2, \ldots, a_{i-1}$, we know that a_j had quotient at least q_j at the time of election. By Proposition 4 applied to the first count and then to the new count, $q_j \geqslant q_i$, and every candidate elected so far in the new count has been elected with a quotient that is at least q_j and hence at least q_i. So if a_i has already been elected in the new count then the conclusion of the Proposition holds for a_i. Since we are supposing that this is not the case, it must be that a_i has not yet been elected. We will consider a_i's quotient and the quota at the start of the next stage, immediately following the election of a_j.

In the first count, a_i was elected with quotient $q_i = v_i/(1+t_i)$, where v_i is the number of ballots that contributed to a_i after a_{i-1}'s election, and t_i is the fractional number of candidates that these ballots had so far elected. These v_i ballots are the ones on which no candidate other than $a_1, \ldots, a_{i-1}$ is preferred to a_i, and so they again contribute to a_i at this point in the new count. So in the new count, a_i now has quotient $\hat{q}_i = (v_i + v'_i)/(1 + \hat{t}_i + \hat{t}'_i)$, where v'_i is the number of ballots contributing to a_i at this point that did not contribute to a_i at the time of a_i's election in the first count, and $\hat{t}_i$ and $\hat{t}'_i$ are the fractional numbers of candidates elected by the original v_i contributors and the new v'_i contributors at this point in the new count. Each of these $v_i + v'_i$ ballots is deemed to have elected either 0 candidates or a number of candidates of the form $1/\hat{q}$, where $\hat{q}$ is the smallest quotient of any elected candidate listed above a_i on that ballot. For all the ballots of this second type, $\hat{q} \geqslant q_j \geqslant q_i$; thus $\hat{t}'_i \leqslant v'_i/q_i$ and $v'_i/\hat{t}'_i \geqslant q_i$. Moreover, for each of the original v_i ballots that is of this second type, the number $\hat{q}$ for that ballot is the smallest of a new set of quotients, each of which is at least as large as the corresponding quotient in the original count, so that if the ballot was electing $1/q$ candidates at the time of a_i's election in the original count then $\hat{q} \geqslant q$ and $1/\hat{q} \leqslant 1/q$; thus $\hat{t}_i \leqslant t_i$. It follows from (1.1) that

$$\hat{q}_i = \frac{v_i + v'_i}{1 + \hat{t}_i + \hat{t}'_i} \geqslant \frac{v_i + v'_i}{1 + t_i + \hat{t}'_i} \geqslant \frac{v_i}{1 + t_i} = q_i. \quad (1.4)$$

Now let us consider the quota. Let v be the number of valid ballots. In the first count, the quota at the time of a_i's election was $Q_i = (v - v_{\mathrm{x}})/(1 + s - t_{\mathrm{x}})$, where v_{x} is the number of *inactive* ballots at the time of a_i's election, and t_{x} is the fractional number of candidates that these ballots have elected. These v_{x} inactive ballots are the ones that contain the name of no candidates other than $a_1, \ldots, a_{i-1}$, and so they are again inactive at this point in the new count. So in the new count, the quota at this point is $\hat{Q}_i = (v - v_{\mathrm{x}} - v'_{\mathrm{x}})/(1 + s - \hat{t}_{\mathrm{x}} - \hat{t}'_{\mathrm{x}})$, where v'_{x} is the number of ballots that were active at the time of a_i's election in the first count but are inactive at this point in the new count, and $\hat{t}_{\mathrm{x}}$ and $\hat{t}'_{\mathrm{x}}$ are the fractional numbers of candidates elected by the original and the new inactive ballots at this point in the new count. By the same argument we used in the previous paragraph to prove that $\hat{t}_i \leqslant t_i$, we can now deduce that $\hat{t}_{\mathrm{x}} \leqslant t_{\mathrm{x}}$. Moreover, by Propositions 2 and 4 and the criterion for election in paragraph 2.5a, every candidate elected so far has been elected with a quotient that is greater than the current quota $\hat{Q}_i$, so that $\hat{t}'_{\mathrm{x}} \leqslant v'_{\mathrm{x}}/\hat{Q}_i$ and $v'_{\mathrm{x}}/t'_{\mathrm{x}} \geqslant \hat{Q}_i$. It follows from (1.1) that

$$\hat{Q}_i = \frac{v - v_{\mathrm{x}} - v'_{\mathrm{x}}}{1 + s - \hat{t}_{\mathrm{x}} - \hat{t}'_{\mathrm{x}}} \leqslant \frac{v - v_{\mathrm{x}}}{1 + s - \hat{t}_{\mathrm{x}}} \leqslant \frac{v - v_{\mathrm{x}}}{1 + s - t_{\mathrm{x}}} = Q_i. \quad (1.5)$$

It follows from (1.4) and (1.5) that $\hat{q}_i \geqslant q_i > Q_i \geqslant \hat{Q}_i$, so that a_i's current quotient is greater than the current quota. Since, by Propositions 2 and 3, the election of other candidates cannot increase the quota nor decrease a_i's quotient, a_i will have a quotient greater than the quota as long as a_i remains hopeful. Thus a_i will eventually be elected, before any exclusions take place, with a quotient that is at least q_i. This contradicts the supposition that the conclusion of the Proposition failed to hold for a_i, and this contradiction completes the proof of the Proposition. □

5 References

[1] V. d'Hondt, *Système Pratique et Raisonné de Représentation Proportionelle [A Practical and Rational System for Proportional Representation]*, Librairie C. Muquardt, Brussels, 1882.

[2] I. D. Hill, B. A. Wichmann and D. R. Woodall, Algorithm 123, Single transferable vote by Meek's method, *The Computer Journal* **30** (1987) 277–281.

[3] R. A. Newland and F. S. Britton, *How to Conduct an Election by the Single Transferable Vote*, Electoral Reform Society of Great Britain and Ireland, London, 1972; 2nd ed. 1976.

[4] E. Phragmén, Proportionella val [Proportional elections], *Svenska Spörsmål* **25**, Stockholm, 1895.

[5] E. Phragmén, Till frågan om en proportionell valmetod [On the question of a proportional election method], *Statsvetenskaplig Tidskrift* **2** (1899), 87–95.

[6] O. Salmi, RE: [EM] D'Hondt without lists, http://groups.yahoo.com/group/election-methods-list/message/10068.

Fuller Disclosure than Intended

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1 Introduction

The full disclosure of preferences in the case of an STV election carries one danger of abuse. That is the potential for a unique preference list to identify a particular voter. Suppose there are 10 candidates in an election. Then there are $10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 = 3\ 628\ 800$ possible complete preference lists as well as a number of incomplete lists. In an electorate of a few tens or hundreds of thousands, it is obvious that the vast majority of the possible preference lists will not be used.

Of the preference lists that are used, they will generally follow some sort of pattern, such as the candidates of one party, followed by the candidates of another party, etc. It will therefore be fairly easy to create a large number of different preference lists that favour a particular candidate (with first preferences), and are most unlikely to be used by any voter.

2 The problem

The full disclosure of preference data facilitates the following fraud: The fraudster bribes or coerces a large number of voters to vote according to an exact preference list that is provided, and is different for each voter. The preference lists provided will be different unlikely sequences, such as the preferred candidate followed by alternate liberals and fascists or conservatives and communists.

Disclosure of the full preference data will then disclose, with a high probability, the voting behaviour of the bribed voters. There may be some false positives, but there will be no false negatives — i.e. if a preference list is missing then it is certain that a bribed voter welched.

3 The solution

One solution has been proposed — that of anonymising the preference data in a similar way to how census data is anonymised. Changes are made to the individual records in such a way as to minimise changes that result to any statistical aggregates an analyst might be interested in. The problem with this is that the statistical analysis of preference data is in such infancy that it is not clear what aggregates should be preserved, or how they might be preserved.

My preferred solution is that prior to disclosure, preference lists should be aggregated by censoring lower preferences until there are at least, say, 3 instances of every preference list to be published. So for example, if there are 10 votes of ABCDEFG then that fact can be published. If there is 1 vote of BCDEFGA, 1 of BCDEFAG and 1 of BCDEGAF then the fact that there were 3 votes of BCDExxx would be published. This would mean that no single individual's vote would be identifiably disclosed.

A New Monotonic and Clone-Independent Single-Winner Election Method

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1 Introduction

In 1997, I proposed to a large number of people who are interested in mathematical aspects of election methods a new method that satisfies Pareto, monotonicity, resolvability, independence of clones, reversal symmetry, Smith-IIA, and Schwartz. This method immediately attracted a lot of attention and very many enthusiastic supporters. Today, this method is promoted e.g. by Diana Galletly [1], Mathew Goldstein [2], Jobst Heitzig [3], Raul Miller, Mike Ossipoff [4], Russ Paielli, Norman Petry, Manoj Srivastava, and Anthony Towns and it is analyzed e.g. in the websites of Blake Cretney [5], Steve Eppley [6], Eric Gorr [7], and Rob LeGrand [8]. Today, this method is taught e.g. by James E. Falk of George Washington University and Thomas K. Yan of Cornell University [9]. In January 2003, the board of Software in the Public Interest (SPI) adopted this method unanimously [10]. In June 2003, the DEBIAN Project adopted this method with 144 against 16 votes [11, 12]. Therefore, a more detailed motivation and explanation of the method is overdue.

There has been some debate about an appropriate name for the method. Some people suggested names like "Beatpath Method", "Beatpath Winner", "Path Voting", "Schwartz Sequential Dropping" (SSD) or "Cloneproof Schwartz Sequential Dropping" (CSSD or CpSSD). However, I prefer the name "Schulze method", not because of academic arrogance, but because the other names do not refer to the method itself but to specific heuristics for implementing it, and so may mislead readers into believing that no other method for implementing it is possible. In my opinion, although it is advantageous to have an intuitive and convincing heuristic, in the end only the properties of the method are relevant.

I have already found some implementations of my method in the internet. Unfortunately, most implementations that I have seen were inefficient because the programmers have not understood the Floyd algorithm so that the implementations had a runtime of $O(N^5)$ although the winners of this method can be calculated in a runtime of $O(N^3)$, where N is the number of candidates.

It is presumed that each voter casts at least a *partial ranking* of all candidates. That means: It is presumed that for each voter V the relation "voter V strictly prefers candidate A to candidate B" is irreflexive, asymmetric, and transitive on the set of candidates. But it is not presumed that each voter casts a *complete ranking*. That means: It is not presumed that this relation is also linear.

Suppose that d[X,Y] is the number of voters who strictly prefer candidate X to candidate Y. Then the *Smith set* is the smallest non-empty set of candidates with d[A,B] $>$ d[B,A] for each candidate A of this set and each candidate B outside this set. *Smith-IIA* (where IIA means Independence from Irrelevant Alternatives) says that adding a candidate who is not in the new Smith set should not change the probability that a given and already running candidate is elected. Smith-IIA implies the majority criterion for solid coalitions and the Condorcet criterion. Unfortunately, compliance with the Condorcet criterion implies violation of other desired criteria like participation [13], later-no-harm, and later-no-help [14].

A *chain from candidate A to candidate B* is an ordered set of candidates C(1),...,C(n) with the following three properties:

1. C(1) is identical to A.
2. C(n) is identical to B.
3. d[C(i),C(i+1)] − d[C(i+1),C(i)] > 0 for each i = 1,...,(n−1).

A *Schwartz winner* is a candidate A who has chains at least to every other candidate B who has a chain to candidate A. The *Schwartz set* is the set of all Schwartz winners. *Schwartz* says that the winner must be a Schwartz winner.

In section 2, the Schulze method is defined. In section 3, well-definedness of this method is proven. In section 4, I present an implementation with a runtime of $O(N^3)$. In section 5, I prove that this method satisfies Pareto, monotonicity, resolvability, independence of clones, and reversal symmetry. From the definition of the Schulze method, it is clear that this method meets Smith-IIA and Schwartz.

Another election method that satisfies Pareto, monotonicity, resolvability, independence of clones, reversal symmetry, Schwartz, and Smith-IIA is Tideman's Ranked Pairs method [15, 16]. However, appendix A demonstrates that the proposed method is not identical with the Ranked Pairs method. Appendix B demonstrates that the proposed method can violate the participation criterion in a very drastic manner. A special provision of the implementation used by SPI and DEBIAN is described in appendix C. Appendix D explains how the proposed method can be interpreted as a method where successively the weakest pairwise defeats are "eliminated." Appendix E presents a concrete example where the proposed method does not find a unique winner.

2 Definition of the Schulze Method

Stage 1: Suppose that d[A,B] is the number of voters who strictly prefer candidate A to candidate B.

A *path from candidate A to candidate B* is an ordered set of candidates C(1),...,C(n) with the following two properties:

1. C(1) is identical to A.
2. C(n) is identical to B.

The *strength* of the path C(1),...,C(n) is min { d[C(i),C(i+1)] − d[C(i+1),C(i)] | i = 1,...,(n−1) }.

Thus a chain from candidate A to candidate B, as defined in the introduction, is simply a path with positive strength.

p[A,B] := max { min { d[C(i),C(i+1)] − d[C(i+1),C(i)] | i = 1,...,(n−1) } | C(1),...,C(n) is a path from candidate A to candidate B }.

In other words: p[A,B] is the strength of the strongest path from candidate A to candidate B.

Candidate A is a *potential winner* if and only if p[A,B] ≥ p[B,A] for every other candidate B.

When p[A,B] > p[B,A], then we say: "Candidate A disqualifies candidate B".

Stage 2: If there is only one potential winner, then this potential winner is the *unique winner*. If there is more than one potential winner, then a *Tie-Breaking Ranking of the Candidates* (TBRC) is calculated as follows:

1. Pick a random ballot and use its rankings; consider ties as unsorted with regard to each other.
2. Continue picking ballots randomly from those that have not yet been picked. When you find one that orders previously unsorted candidates, use the ballot to sort them. Do not change the order of the already sorted.
3. If you go through all ballots, and some candidates are still not sorted, order them randomly.

The winner is that potential winner who is ranked highest in this TBRC.

3 Well-Definedness

On first view, it is not clear whether the Schulze method is well defined. It seems to be possible that candidates disqualify each other in such a manner that there is no candidate A with p[A,B] ≥ p[B,A] for every other candidate B. However, the following proof demonstrates that path defeats are transitive. That means: When candidate A disqualifies candidate B and when candidate B disqualifies candidate C, then also candidate A disqualifies candidate C.

Claim: (p[A,B] > p[B,A] and p[B,C] > p[C,B]) ⇒ p[A,C] > p[C,A].

Proof: Suppose
(1) p[A,B] > p[B,A] and
(2) p[B,C] > p[C,B].

The following statements are valid:
(3) min { p[A,B]; p[B,C] } $\leq$ p[A,C].
(4) min { p[A,C]; p[C,B] } $\leq$ p[A,B].
(5) min { p[B,A]; p[A,C] } $\leq$ p[B,C].
(6) min { p[B,C]; p[C,A] } $\leq$ p[B,A].
(7) min { p[C,A]; p[A,B] } $\leq$ p[C,B].
(8) min { p[C,B]; p[B,A] } $\leq$ p[C,A].

For example: If min { p[A,B]; p[B,C] } was strictly larger than p[A,C], then this would be a contradiction to the definition of p[A,C] since there would be a route from candidate A to candidate C via candidate B with a strength of more than p[A,C]; and if this route was not itself a path (because it passed through some candidates more than once) then some subset of its links would form a path from candidate A to candidate C with a strength of more than p[A,C].

Case 1: Suppose
(9a) p[A,B] $\geq$ p[B,C].
Combining (2) and (9a) gives:
(10a) p[A,B] $>$ p[C,B].
Combining (7) and (10a) gives:
(11a) p[C,A] $\leq$ p[C,B].
Combining (3) and (9a) gives:
(12a) p[B,C] $\leq$ p[A,C].
Combining (11a), (2), and (12a) gives:
(13a) p[C,A] $\leq$ p[C,B] $<$ p[B,C] $\leq$ p[A,C].

Case 2: Suppose
(9b) p[A,B] $<$ p[B,C].
Combining (1) and (9b) gives:
(10b) p[B,C] $>$ p[B,A].
Combining (6) and (10b) gives:
(11b) p[C,A] $\leq$ p[B,A].
Combining (3) and (9b) gives:
(12b) p[A,B] $\leq$ p[A,C].
Combining (11b), (1), and (12b) gives:
(13b) p[C,A] $\leq$ p[B,A] $<$ p[A,B] $\leq$ p[A,C].

Therefore, the relation defined by p[A,B] $>$ p[B,A] is transitive.

4 Implementation

The strength of the strongest path p[i,j] from candidate i to candidate j can be calculated with the Floyd algorithm [17]. The runtime to calculate the strengths of all paths is $O(N^3)$. It cannot be said frequently enough that the order of the indices in the triple-loop of the Floyd algorithm is **not** irrelevant.

Input: d[i,j] with i $\neq$ j is the number of voters who strictly prefer candidate i to candidate j.

Output: "w[i] = true" means that candidate i is a potential winner. "w[i] = false" means that candidate i is not a potential winner.

```
for i := 1 to N do
for j := 1 to N do
  if ( i ≠ j ) then
    p[i,j] := d[i,j] − d[j,i] ;

for i := 1 to N do
for j := 1 to N do
  if ( i ≠ j ) then
    for k := 1 to N do
      if ( i ≠ k ) then
        if ( j ≠ k ) then
          {
          s := min { p[j,i], p[i,k] } ;
          if ( p[j,k] < s ) then
            p[j,k] := s ;
          }

for i := 1 to N do
  {
  w[i] := true ;
  for j := 1 to N do
    if ( i ≠ j ) then
      if ( p[j,i] > p[i,j] ) then
        w[i] := false ;
  }
```

5 Properties

5.1 Pareto

Pareto says that when no voter strictly prefers candidate B to candidate A and at least one voter strictly prefers candidate A to candidate B then candidate B must not be elected.

The Schulze method meets Pareto.

Proof: Suppose no voter strictly prefers candidate B to candidate A and at least one voter strictly prefers candidate A to candidate B. Then d[A,B] $>$ 0 and d[B,A] = 0.

Case 1: If BA is already the strongest path from candidate B to candidate A, then p[B,A] = d[B,A] − d[A,B] $<$ 0. Therefore, candidate A disqualifies candidate B because p[A,B] $\geq$ d[A,B] − d[B,A] $>$ 0, so that p[A,B] $>$ p[B,A].

Case 2: Suppose that B,C(1),…,C(n),A is the strongest path from candidate B to candidate A. As every voter who strictly prefers candidate B to candidate C(1) also necessarily strictly prefers candidate A to candidate C(1), we get d[A,C(1)] $\geq$ d[B,C(1)]. As every voter who strictly prefers candidate C(1) to candidate A also necessarily strictly prefers candidate C(1) to candidate B, we get d[C(1),B] $\geq$ d[C(1),A]. Therefore, d[A,C(1)] − d[C(1),A] $\geq$ d[B,C(1)] − d[C(1),B]. For the same reason, we get d[C(n),B] − d[B,C(n)] $\geq$ d[C(n),A] − d[A,C(n)]. Therefore, the path A,C(1),…,C(n),B is at least as strong as the path B,C(1),…,C(n),A. In so far as B,C(1),…,C(n),A is the strongest path from candidate B to candidate A by presumption, we get p[A,B] $\geq$ p[B,A].

Suppose that candidate B is a potential winner. Then also candidate A is a potential winner.

Proof: Suppose that B,C(1),…,C(n),X is the strongest path from candidate B to candidate X. Then, A,C(1),...,C(n),X is a path, but not necessarily the strongest path, from candidate A to candidate X with at least the same strength because d[A,C(1)] − d[C(1),A] $\geq$ d[B,C(1)] − d[C(1),B]. Therefore, p[A,X] $\geq$ p[B,X] for every candidate X other than candidate A or candidate B. Suppose that X,C(1),…,C(n),A is the strongest path from candidate X to candidate A. Then, X,C(1),...,C(n),B is a path, but not necessarily the strongest path, from candidate X to candidate B with at least the same strength because d[C(n),B] − d[B,C(n)] $\geq$ d[C(n),A] − d[A,C(n)]. Therefore, p[X,B] $\geq$ p[X,A] for every candidate X other than candidate A or candidate B.

Since candidate B is a potential winner, p[B,X] $\geq$ p[X,B] for every other candidate X. With p[A,X] $\geq$ p[B,X], p[B,X] $\geq$ p[X,B], and p[X,B] $\geq$ p[X,A], we get p[A,X] $\geq$ p[X,A] for every other candidate X. Therefore, also candidate A is a potential winner.

Therefore, when no voter strictly prefers candidate B to candidate A and at least one voter strictly prefers candidate A to candidate B then when candidate B is a potential winner also candidate A is a potential winner. Therefore, candidate B cannot be elected at stage 1 of the Schulze method. Candidate B cannot be elected at stage 2, either, since candidate A is necessarily ranked above candidate B in the TBRC.

5.2 Monotonicity

Monotonicity says that when some voters rank candidate A higher without changing the order in which they rank the other candidates relatively to each other then the probability that candidate A is elected must not decrease.

The Schulze method meets monotonicity.

Proof: Suppose candidate A was a potential winner. Then p_{old}[A,B] $\geq$ p_{old}[B,A] for every other candidate B.

Part 1: Suppose some voters rank candidate A higher without changing the order in which they rank the other candidates. Then d_{new}[A,X] $\geq$ d_{old}[A,X] and d_{new}[X,A] $\leq$ d_{old}[X,A] for every other candidate X. d_{new}[X,Y] = d_{old}[X,Y] when neither candidate X nor candidate Y is identical to candidate A. Therefore d_{new}[A,X] − d_{new}[X,A] $\geq$ d_{old}[A,X] − d_{old}[X,A] for every other candidate X. And d_{new}[X,Y] − d_{new}[Y,X] = d_{old}[X,Y] − d_{old}[Y,X] when neither candidate X nor candidate Y is identical to candidate A. For every candidate B other than candidate A the value p[A,B] can only increase but not decrease with d[A,X] − d[X,A] since only AX but not XA can be in the strongest path from candidate A to candidate B and the value p[B,A] can only decrease but not increase with d[A,X] − d[X,A] since only XA but not AX can be in the strongest path from candidate B to candidate A. Therefore p_{new}[A,B] $\geq$ p_{old}[A,B] and p_{new}[B,A] $\leq$ p_{old}[B,A]. Therefore p_{new}[A,B] $\geq$ p_{new}[B,A] so that candidate A is still a potential winner.

Part 2: Suppose that candidate E is not identical to candidate A. It remains to be proven that when candidate E was not a potential winner before then he is still not a potential winner. Suppose that candidate E was not a potential winner. Then there must have been a candidate F other than candidate E with

(1) p_{old}[F,E] $>$ p_{old}[E,F].

Then, of course, also p_{new}[F,E] $>$ p_{new}[E,F] is valid unless XA was a weakest link in the strongest path from candidate F to candidate E and/or AY was the weakest link in the strongest path from candidate E to candidate F. Without loss of generality, we can presume that candidate F is not identical to candidate A and that

(2) p_{old}[A,E] = p_{old}[E,A]

because otherwise with p_{old}[A,E] $>$ p_{old}[E,A] we would immediately get p_{new}[A,E] $>$ p_{new}[E,A] (because of the considerations in Part 1) so that we would immediately get that candidate E is still not a potential winner. Since candidate A was a potential winner, we get

(3) p_{old}[A,F] $\geq$ p_{old}[F,A].

The following statements are valid for the same reason as in section 3:

(4) min { p_{old}[A,E]; p_{old}[E,F] } $\leq$ p_{old}[A,F].
(5) min { p_{old}[A,F]; p_{old}[F,E] } $\leq$ p_{old}[A,E].
(6) min { p_{old}[E,A]; p_{old}[A,F] } $\leq$ p_{old}[E,F].
(7) min { p_{old}[E,F]; p_{old}[F,A] } $\leq$ p_{old}[E,A].
(8) min { p_{old}[F,A]; p_{old}[A,E] } $\leq$ p_{old}[F,E].
(9) min { p_{old}[F,E]; p_{old}[E,A] } $\leq$ p_{old}[F,A].

Case 1: Suppose XA was a weakest link in the strongest path from candidate F to candidate E. Then

(10a) p_{old}[F,E] = p_{old}[F,A] and
(11a) p_{old}[A,E] $\geq$ p_{old}[F,E].
Now (3), (10a), and (1) give
(12a) p_{old}[A,F] $\geq$ p_{old}[F,A] = p_{old}[F,E] $>$ p_{old}[E,F],
while (2), (11a), and (1) give
(13a) p_{old}[E,A] = p_{old}[A,E] $\geq$ p_{old}[F,E] $>$ p_{old}[E,F].
But (12a) and (13a) together contradict (6).

Case 2: Suppose AY was the weakest link in the strongest path from candidate E to candidate F. Then

(10b) p_{old}[E,F] = p_{old}[A,F] and
(11b) p_{old}[E,A] $>$ p_{old}[E,F].
Now (11b), (10b), and (3) give
(12b) p_{old}[E,A] $>$ p_{old}[E,F] = p_{old}[A,F] $\geq$ p_{old}[F,A],
while (1), (10b), and (3) give
(13b) p_{old}[F,E] $>$ p_{old}[E,F] = p_{old}[A,F] $\geq$ p_{old}[F,A].
But (12b) and (13b) together contradict (9).

Conclusion: When some voters rank candidate A higher without changing the order in which they rank the other candidates relatively to each other, then (a) when candidate A was a potential winner candidate A is still a potential winner and (b) every other candidate E who was not a potential winner is still not a potential winner and (c) candidate A can only increase in the TBRC while the positions of the other candidates are not changed relatively to each other. Therefore, the probability that candidate A is elected cannot decrease.

5.3 Resolvability

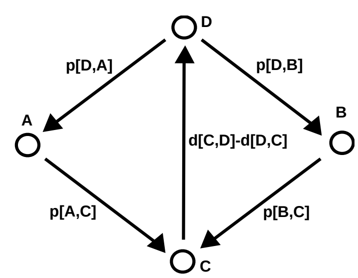
Resolvability says that at least in those cases in which there are no pairwise ties and there are no pairwise defeats of equal strength the winner must be unique.

The Schulze method meets resolvability.

Proof: Suppose that there is no unique winner. Suppose that candidate A and candidate B are potential winners. Then:

(1) p[A,B] = p[B,A].

Suppose that there are no pairwise ties and that there are no pairwise defeats of equal strength. Then p[A,B] = p[B,A] means that the weakest link in the strongest path from candidate A to candidate B and the weakest link in the strongest path from candidate B to candidate A must be the same link, say CD. Then this situation looks as follows:



As the weakest link of the strongest path from candidate B to candidate A is CD, we get:
(2) p[D,A] $>$ p[B,A].
As the weakest link of the strongest path from candidate A to candidate B is CD, we get:
(3) p[A,D] = p[A,B].
With (2), (1), and (3) we get:
(4) p[D,A] $>$ p[B,A] = p[A,B] = p[A,D] which contradicts the presumption that candidate A is a potential winner.

5.4 Independence of Clones

An election method is *independent of clones* if the following holds:

Suppose that candidate D and candidate E are two different candidates.

1. Suppose (a) that there is at least one voter who either strictly prefers candidate D to candidate E or strictly prefers candidate E to candidate D or (b) that candidate D is elected with zero probability.
2. Suppose that candidate D is replaced by a set of candidates D(1),...,D(m) in such a manner that for every candidate D(i) in this set, for every candidate F outside this set, and for every voter V the following two statements are valid:
   a) V strictly preferred D to F ⇔ V strictly prefers D(i) to F.
   b) V strictly preferred F to D ⇔ V strictly prefers F to D(i).

Then the probability that candidate E is elected must not change.

The Schulze method is independent of clones.

Proof: Suppose that candidate D is replaced by a set of candidates D(1),…,D(m) in the manner described above. Then d_{new}[A,D(i)] = d_{old}[A,D] for every candidate A outside the set D(1),…,D(m) and for every i = 1,…,m. And d_{new}[D(i),B] = d_{old}[D,B] for every candidate B outside the set D(1),…,D(m) and for every i = 1,…,m.

(1) Case 1: Suppose that the strongest path C(1),…,C(n) from candidate A to candidate B did not contain candidate D. Then C(1),…,C(n) is still a path from candidate A to candidate B with the same strength. Therefore: p_{new}[A,B] $\geq$ p_{old}[A,B].

Case 2: Suppose that the strongest path C(1),…,C(n) from candidate A to candidate B contained candidate D. Then C(1),…,C(n) with D replaced by an arbitrarily chosen candidate D(i) is still a path from candidate A to candidate B with the same strength. Therefore: p_{new}[A,B] $\geq$ p_{old}[A,B].

(2) Case 1: Suppose that the strongest path C(1),…,C(n) from candidate A to candidate B does not contain candidates of the set D(1),…,D(m). Then C(1),…,C(n) was a path from candidate A to candidate B with the same strength. Therefore: p_{old}[A,B] $\geq$ p_{new}[A,B].

Case 2: Suppose that the strongest path C(1),…,C(n) from candidate A to candidate B contains some candidates of the set D(1),…,D(m). Then C(1),…,C(n) where the part of this path from the first occurrence of a candidate of the set D(1),…,D(m) to the last occurrence of a candidate of the set D(1),…,D(m) is replaced by candidate D was a path from candidate A to candidate B with at least the same strength. Therefore: p_{old}[A,B] $\geq$ p_{new}[A,B].

With (1) and (2), we get: p_{new}[A,B] = p_{old}[A,B].

When we set A $\equiv$ D in (1) and (2), we get: p_{new}[D(i),B] = p_{old}[D,B] for every candidate B outside the set D(1),…,D(m) and for every i = 1,…,m.

When we set B $\equiv$ D in (1) and (2), we get: p_{new}[A,D(i)] = p_{old}[A,D] for every candidate A outside the set D(1),…,D(m) and for every i = 1,…,m.

Suppose candidate A, who is not identical to candidate D, was a potential winner, then p_{old}[A,B] $\geq$ p_{old}[B,A] for every other candidate B; because of the above considerations we get p_{new}[A,B] $\geq$ p_{new}[B,A] for every other candidate B; therefore, candidate A is still a potential winner. Suppose candidate B, who is not identical to candidate D, was not a potential winner, then p_{old}[B,A] $<$ p_{old}[A,B] for at least one other candidate A; because of the above considerations we get p_{new}[B,A] $<$ p_{new}[A,B] for at least this other candidate A; therefore, candidate B is still not a potential winner.

Presumption 1 in the definition of independence of clones guarantees that at least in those situations in which the TBRC has to be used to choose from the candidates D(1),…,D(m),E (a) candidate E is ranked above each of the candidates D(1),...,D(m) when he was originally ranked above candidate D. (b) candidate E is ranked below each of the candidates D(1),…,D(m) when he was originally ranked below candidate D. Therefore, replacing candidate D by a set of candidates D(1),…,D(m) can neither change whether candidate E is a potential winner nor, when the TBRC has to be used, where this candidate is ranked in the TBRC.

5.5 Reversal Symmetry

Reversal symmetry says that when candidate A is the unique winner then when the individual preferences of each voter are inverted then candidate A must not be elected.

The Schulze method meets reversal symmetry.

Proof: Suppose candidate A was the unique winner. Then there must have been at least one other candidate B with p_{old}[A,B] $>$ p_{old}[B,A]. (Since the relation defined by p[X,Y] $>$ p[Y,X] is transitive there must have been at least one candidate B other than candidate A with p[B,E] $\geq$ p[E,B] for every candidate E other than candidate A or candidate B. Since candidate A was the unique winner and since no candidate other than candidate A has disqualified candidate B, candidate A must have disqualified candidate B, i.e. p_{old}[A,B] > p_{old}[B,A].)

When the individual preferences of each voter are inverted then d_{new}[Y,X] = d_{old}[X,Y] for each pair XY of candidates. When C(1),...,C(n) was a path from candidate X to candidate Y of strength Z then C(n),...,C(1) is a path from candidate Y to candidate X of strength Z. Therefore, p_{new}[Y,X] = p_{old}[X,Y] for each pair XY of candidates. Therefore, p_{new}[B,A] > p_{new}[A,B] so that candidate B disqualifies candidate A.

6 References

[1] http://www.chiark.greenend.org.uk/∼galletly/university/speech2.pdf

[2] http://condorcet-dd.sourceforge.net

[3] http://www.groucho.info

[4] http://www.electionmethods.org

[5] http://www.condorcet.org

[6] http://www.alumni.caltech.edu/∼seppley

[7] http://www.ericgorr.net/condorcet

[8] http://userfs.cec.wustl.edu/∼rhl1/rbvote/desc.html

[9] http://www.cs.cornell.edu/Courses/cs100m/2001SP/projects.htm

[10] http://www.spi-inc.org/corporate/resolutions/resolution-2003-01-06.wta.1

[11] http://www.debian.org/devel/ constitution

[12] http://www.debian.org/vote/2003/ vote_0002

[13] Hervé Moulin, Condorcet's Principle Implies the No Show Paradox, *Journal of Economic Theory*, vol. 45, p. 53-64, 1988

[14] Douglas R. Woodall, Monotonicity of single-seat preferential election rules, *Discrete Applied Mathematics*, vol. 77, p. 81-98, 1997

[15] T. Nicolaus Tideman, Independence of Clones as a Criterion for Voting Rules, *Social Choice and Welfare*, vol. 4, p. 185-206, 1987

[16] Thomas M. Zavist, T. Nicolaus Tideman, Complete Independence of Clones in the Ranked Pairs Rule, *Social Choice and Welfare*, vol. 6, p. 167-173, 1989

[17] Robert W. Floyd, Algorithm 97 (Shortest Path), *Communications of the ACM*, vol. 5, p. 345-350, 1962

[18] Joaquín Pérez, The Strong No Show Paradoxes are a common flaw in Condorcet voting correspondences, *Social Choice and Welfare*, vol. 18, p. 601-616, 2001

[19] M.J.A.N. de Condorcet, *Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix*, Imprimerie Royale, Paris, 1785

[20] Peyton Young, Condorcet's Theory of Voting, *American Political Science Review*, vol. 82, p. 1231-1244, 1988

[21] Clarence G. Hoag, George H. Hallett, *Proportional Representation*, MacMillan Company, New York, 1926

A Tideman's Ranked Pairs Method

Tideman's Ranked Pairs method [15, 16] is very similar to my method in so far as both methods meet Pareto, monotonicity, resolvability, independence of clones, reversal symmetry, Schwartz and Smith-IIA. However, the following example demonstrates that these methods are not identical.

Example:

3	ACDB
5	ADBC
4	BACD
5	BCDA
2	CADB
5	CDAB
2	DABC
4	DBAC

The matrix d[i,j] of pairwise defeats looks as follows:

	A	B	C	D
A	—	17	18	14
B	13	—	20	9
C	12	10	—	19
D	16	21	11	—

The matrix p[i,j] of the path strengths looks as follows:

	A	B	C	D
A	—	6	6	6
B	2	—	10	8
C	2	8	—	8
D	2	12	10	—

Candidate A is the unique Schulze winner because candidate A is the unique candidate with p[A,X] ≥ p[X,A] for every other candidate X.

Tideman suggests to take successively the strongest pairwise defeat and to lock it if it does not create a directed cycle with already locked pairwise defeats or to skip it if it would create a directed cycle with already locked pairwise defeats. The winner of the Ranked Pairs method is that candidate X who wins each pairwise comparison which is locked and in which candidate X is involved.

Tideman's Ranked Pairs method locks D > B. Then it locks B > C. Then it skips C > D since it would create a directed cycle with the already locked defeats D > B and B > C. Then it locks A > C. Then it locks A > B. Then it locks D > A. Thus, the Ranked Pairs winner is candidate D.

B The Participation Criterion

The *participation* criterion says that adding a set of identical ballots on which candidate A is strictly preferred to candidate B should not change the winner from candidate A to candidate B. Moulin [13] proved that the Condorcet criterion and the participation criterion are incompatible. Pérez [18] demonstrated that most Condorcet methods can violate the participation criterion in a very drastic manner. That means: It can happen that adding a set of identical ballots on which candidate A is strictly preferred to every other candidate changes the winner from candidate A to another candidate or that adding a set of identical ballots on which every other candidate is strictly preferred to candidate B changes the winner from another candidate to candidate B. The following example demonstrates that also the Schulze method can violate the participation criterion in a very drastic manner. (The basic idea for this example came from Blake Cretney.)

Example:

4	ABCDEF
2	ABFDEC
4	AEBFCD
2	AEFBCD
2	BFACDE
2	CDBEFA
4	CDBFEA
12	DECABF
8	ECDBFA
10	FABCDE
6	FABDEC
4	FEDBCA

The matrix d[i,j] of pairwise defeats looks as follows:

	A	B	C	D	E	F
A	—	40	30	30	30	24
B	20	—	34	30	30	38
C	30	26	—	36	22	30
D	30	30	24	—	42	30
E	30	30	38	18	—	32
F	36	22	30	30	28	—

The matrix p[i,j] of the path strengths looks as follows:

	A	B	C	D	E	F
A	—	20	8	8	8	16
B	12	—	8	8	8	16
C	4	4	—	12	12	4
D	4	4	16	—	24	4
E	4	4	16	12	—	4
F	12	12	8	8	8	—

Candidate A is the unique winner since he is the only candidate with p[A,X] ≥ p[X,A] for every other candidate X. However, when 3 AEFCBD ballots are added then the matrix d[i,j] of pairwise defeats looks as follows:

	A	B	C	D	E	F
A	—	43	33	33	33	27
B	20	—	34	33	30	38
C	30	29	—	39	22	30
D	30	30	24	—	42	30
E	30	33	41	21	—	35
F	36	25	33	33	28	—

The matrix p[i,j] of the path strengths looks as follows:

	A	B	C	D	E	F
A	—	23	5	5	5	13
B	9	—	5	5	5	13
C	7	7	—	15	15	7
D	7	7	19	—	21	7
E	7	7	19	15	—	7
F	9	9	5	5	5	—

Now, candidate D is the unique winner since he is the only candidate with p[D,X] ≥ p[X,D] for every other candidate X. Thus the 3 AEFCBD voters change the winner from candidate A to candidate D.

C A Special Provision of the Implementation used by SPI and DEBIAN

There has been some debate about how to measure the strength of a pairwise defeat when it is presumed that on the one side each voter has a sincere complete ranking of all candidates, but on the other side some voters vote only a partial ranking because of strategical considerations. I suggest that then the strength of a pairwise defeat should be measured primarily by the absolute number of votes for the winner of this pairwise defeat and secondarily by the margin of this pairwise defeat. The purpose of this provision is to give an additional incentive to the voters to give different preferences to candidates to which the voters would have given the same preference because of strategical considerations otherwise.

The resulting version of this method is used by SPI and DEBIAN because (a) here the number of candidates is usually very small and the voters are usually well-informed about the different candidates so that it can be presumed that each voter has a sincere complete ranking of all candidates and (b) here the number of voters is usually very small and the voters are usually well-informed about the opinions of the other voters so that the incentive to cast only a partial ranking because of strategical considerations is large.

The resulting version still satisfies Pareto, monotonicity, resolvability, independence of clones, reversal symmetry, Smith-IIA, and Schwartz. When each voter casts a complete ranking then this version is identical to the version defined in section 2. I suggest that in the general case the version as defined in section 2 should be used. Only in situations similar to the above described situation in SPI and DEBIAN, the version as defined in this appendix should be used.

When the strength of a pairwise defeat is measured primarily by p1 (= the absolute number of votes for the winner of this pairwise defeat) and secondarily by p2 (= the margin of this pairwise defeat), then a possible implementation looks as follows:

Input: d[i,j] with $i \neq j$ is the number of voters who strictly prefer candidate i to candidate j.

Output: "w[i] = true" means that candidate i is a potential winner. "w[i] = false" means that candidate i is not a potential winner.

```
for i := 1 to N do
for j := 1 to N do
  if ( i ≠ j ) then
    {
    p2[i,j] := d[i,j] − d[j,i] ;
    if ( d[i,j] > d[j,i] ) then
      p1[i,j] := d[i,j] ;
    if ( d[i,j] ≤ d[j,i] ) then
      p1[i,j] := −1 ;
    }

for i := 1 to N do
for j := 1 to N do
  if ( i ≠ j ) then
    for k := 1 to N do
      if ( i ≠ k ) then
        if ( j ≠ k ) then
          {
          s := min { p1[j,i], p1[i,k] } ;
          t := min { p2[j,i], p2[i,k] } ;
          if ( ( p1[j,k] < s ) or ( ( p1[j,k] = s ) and
              ( p2[j,k] < t ) ) ) then
            {
            p1[j,k] := s ;
            p2[j,k] := t ;
            }
          }

for i := 1 to N do
  {
  w[i] := true ;
  for j := 1 to N do
    if ( i ≠ j ) then
    if ( ( p1[j,i] > p1[i,j] ) or ( ( p1[j,i] = p1[i,j] ) and
          ( p2[j,i] > p2[i,j] ) ) ) then
      w[i] := false ;
  }
```

D The Schwartz Set Heuristic

Another way of looking at the proposed method is to interpret it as a method where successively the weakest pairwise defeats are "eliminated". The formulation of this method then becomes very similar to Condorcet's original wordings.

Condorcet writes [19] p. 126: "Create an opinion of those $N(N\text{-}1)/2$ propositions that win most of the votes. If this opinion is one of the $N!$ possible then consider as elected that subject to which this opinion agrees with its preference. If this opinion is one of the $(2^{(N(N-1)/2)}) - (N!)$ impossible opinions then eliminate of this impossible opinion successively those propositions that have a smaller plurality and accept the resulting opinion of the remaining propositions."

In short, Condorcet suggests that the weakest pairwise defeats should be eliminated successively until the remaining pairwise defeats form a ranking of the candidates. The problem with Condorcet's proposal is that it is not quite clear what it means to "eliminate" a pairwise defeat (especially in so far as when one successively eliminates the weakest pairwise defeat that is in a directed cycle of not yet eliminated pairwise defeats until there are no directed cycles of non-eliminated pairwise defeats any more then the remaining pairwise defeats usually do not complete to a unique ranking [20]). It is clear what it means when a candidate is "eliminated"; this candidate is treated as if he has never stood. But what does it mean when the pairwise defeat A $>$ B is "eliminated" although candidate A and candidate B are still potential winners?

A possible interpretation would be to say that the "elimination" of a pairwise defeat is its replacing by a pairwise tie. However, when this interpretation is being used then the Smith set, as defined in the Introduction, can only grow but not shrink at each stage. But when the Schwartz set, as defined in the Introduction, is being used, then the number of candidates decreases continuously. With the concept of the Schwartz set the Schulze method can be described in a very concise manner:

Step 1: Calculate the Schwartz set and eliminate all those candidates who are not in the Schwartz set. Eliminated candidates stay eliminated.

If there is still more than one candidate and there are still pairwise comparisons between non-eliminated candidates that are not pairwise ties: Go to Step 2.

If there is still more than one candidate, but all pairwise comparisons between non-eliminated candidates are pairwise ties, then all remaining candidates are *potential winners*: Go to Step 3.

If there is only one candidate, then this candidate is the *unique winner*.

Step 2: The weakest pairwise defeat between two non-eliminated candidates is replaced by a pairwise tie. Pairwise comparisons that have been replaced by pairwise ties stay replaced by pairwise ties.

In the version in section 4, the *weakest pairwise defeat* is that defeat where |d[i,j] $-$ d[j,i]| is minimal.

In the version in appendix C, the *weakest pairwise defeat* is that defeat where the number of votes for the winner of this pairwise defeat is minimal or —if there is more than one pairwise defeat where the number of votes for the winner is minimal— of all those pairwise defeats where the number of votes for the winner is minimal that pairwise defeat where the number of votes for the loser of this pairwise defeat is maximal.

If the weakest pairwise defeat between non-eliminated candidates is not unique, then all weakest pairwise defeats between non-eliminated candidates are replaced by pairwise ties simultaneously. Go to Step 1.

Step 3: The TBRC is calculated as described in section 2. The winner is that potential winner who is ranked highest in this TBRC.

E An Example without a Unique Winner

Example [21], p. 502:

3	ABCD
2	DABC
2	DBCA
2	CBDA

The matrix d[i,j] of pairwise defeats looks as follows:

	A	B	C	D
A	—	5	5	3
B	4	—	7	5
C	4	2	—	5
D	6	4	4	—

The matrix p[i,j] of the path strengths looks as follows:

	A	B	C	D
A	—	1	1	1
B	1	—	5	1
C	1	1	—	1
D	3	1	1	—

Candidate X is a potential winner if and only if p[X,Y] $\geq$ p[Y,X] for every other candidate Y. Therefore, candidate B and candidate D are potential winners.

When the Schwartz set heuristic is being used then at the first stage the Schwartz set is calculated. The pairwise defeats are A $>$ B, A $>$ C, B $>$ C, B $>$ D, C $>$ D, and D $>$ A. Hence, the Schwartz set is: A, B, C, and D. At the second stage, the weakest pairwise defeat that is not a pairwise tie between candidates who have not yet been eliminated is replaced by a pairwise tie. The weakest pairwise defeats are A $>$ B, A $>$ C, B $>$ D, and C $>$ D each with a strength of 5:4. All these pairwise defeats are replaced by pairwise ties simultaneously. The remaining pairwise defeats are B $>$ C and D $>$ A. Hence, the new Schwartz set is: B and D. Since there are now no pairwise defeats between candidates who have not yet been eliminated, the algorithm stops and candidate B and candidate D are the winners.

Since 5 voters strictly prefer candidate B to candidate D and 4 voters strictly prefer candidate D to candidate B, candidate B is ranked higher than candidate D in the TBRC with a probability of 5/9 and candidate D is ranked higher than candidate B in the TBRC with a probability of 4/9. Therefore, the winner of the Schulze method is candidate B with a probability of 5/9 and candidate D with a probability of 4/9.

Calculation of Transfer Values — Proposal for STV-PR Rules for Local Government Elections in Scotland

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1 Introduction

The Local Governance (Scotland) Bill [1] will make provision for future local government elections in Scotland to be by the Single Transferable Vote. Those responsible for drafting the legislation have indicated that they do not intend simply to copy the legislation used for the comparable STV elections in Northern Ireland. They believe they can express some points in the counting procedure more clearly. Thus we have a "painless" opportunity to consider some other changes that might usefully be incorporated at the same time. I suggest one of these should be the calculation of transfer values.

2 Precision of calculation

Some discussion in the Election Methods web group [2] prompted me to look in some depth at the calculation of transfer values in STV-PR. The discussion was started by a reference to Wichmann's review [3] of the ERS97 Rules [4]. Wichmann made a number of points about transfer values, starting with what I would call "apparent precision", but going into the arithmetical realities of the truncated calculations adopted in ERS97 and other sets of rules based on Newland and Britton 1972 [5], including those currently used in Northern Ireland. Wichmann's proposal to give results with an actual accuracy of 0.01 votes was to compute transfer values to [(number of digits in total votes) + 1].

Another member of the EM web group drew attention to the procedures of the Australian Electoral Commission [6]. The AEC calculates transfer values to eight decimal places and then truncates as shown in the example on their website. This requirement to calculate to eight decimal places is not specified in any Australian legislation, but only in the AEC's internal working documents [7]. The relevant law [8] makes no reference to the accuracy or precision for any of the STV calculations. The AEC adopted eight decimal places because that was the limit of the desktop calculators available at the time they framed that working rule [7].

The AEC example shows that while they calculate the transfer value of a ballot paper to eight decimal places (8dp) and then use that 8dp result to calculate the transfer values of the votes being transferred, they truncate the candidates' transferred votes to integer values. They do not show decimal parts of a vote anywhere on their result sheets. This truncation to integer values might seem perverse, but does not result in the loss of significant numbers of votes.

In the AEC example there is a surplus of 992,137 votes carried on 1,518,178 papers, of which one candidate receives 1,513,870 papers. The AEC calculation shows an 8dp truncated transfer value of 0.65350505 for each paper. This results in a candidate integer truncated transfer vote of 989,321. The "full" calculation with the 8dp transfer value would have been 989321.69, so they have lost only 0.69 of a vote by integer truncation. This amounts to only 0.000131% of the quota. Had the transfer value been calculated to 15dp (limit of numerical precision for Microsoft Excel 2002), the loss by integer truncation of the votes transferred would have been only 0.700215653, amounting to 0.000133% of the quota.

In contrast, using the ERS/NI rules and calculating the same example to only two decimal places and then truncating, gives a transfer value of 0.65, and a candidate transfer vote of 984,015.50. In this case there would be a loss of 5,306.20 votes from the "true" transfer value, amounting to 1.01% of the quota.

3 Examples from elections

For practical examples I have looked at the immediately available results from the Australian Federal Senate elections in 1998 [9] and the Northern Ireland Assembly elections of 1998 [10]. To make sure there were no complications in the calculations, I looked only at separate transfers arising from the surpluses of candidates whose first preference votes exceeded the quota, i.e. who were elected at stage 1. The relevant figures are in the Tables 1 and 2. In the Australian results they show "non-transferable votes" separately for "exhausted ballots" and for "lost by fraction", ie due to truncation.

The losses arising from truncation are expressed as percentages of the quotas for the relevant elections because this offers the most valid basis for comparisons among the different elections. The results are sorted in ascending order by the size of these percentages. The losses in the Australian transfers range from 0.0043% to 0.032%. In only six of those 14 transfers did the loss exceed 0.01% of the quota. The losses in the Northern Ireland transfers range from 0.10% to 1.36%. In five of those 23 transfers the loss exceeded 1.0% of the quota.

The size of the loss in any individual transfer will depend on just how the calculation tumbles out as that will determine the size of the fraction truncated. For example, in the Newry and Armagh election the transfer value was 0.43 (excluding 222 exhausted papers), leading to a loss of 0.0077245 votes on every one of the 13,360 papers actually transferred. In the Australian elections the losses are increased by the large numbers of candidates who stand and to whom transfers are made.

4 Proposal for change

It now seems clear to me that when the STV rules were formalised for Newland and Britton and the Northern Ireland STV regulations in 1972, there was a confusion of two objectives. It is illogical to calculate transfer values to only two decimal places if candidates' votes are to be recorded to of 0.01 of a vote. This approach was probably taken because the 'Senatorial Rules' [11], devised to remove the element of chance when selecting full value ballot papers for the transfer of surpluses, had given each valid ballot paper a value of one hundred before any calculations were done.

For public elections, with large numbers of electors, there is no intrinsic merit in recording candidates' votes with a precision greater than one vote, provided that does not result in the loss of significant numbers of votes. For elections with small numbers of electors (quota less than 100), there may be a benefit in recording candidates' votes with greater precision, perhaps to 0.01 of a vote. Whatever level of precision is required in the recorded vote, calculating transfer values of ballot papers to only two places of decimals is not consistent with that reported precision. There may be a theoretical case for varying the numbers of decimal places in the calculation according the magnitude of the numbers of votes, but the practical approach of the AEC has been shown to give very satisfactory results.

The AEC adopted eight decimal places for the calculation of transfer values because that was the capacity of the desktop calculators available at the time. Most currently available electronic calculators (hand-held and desktop models) display eight decimal digits, i.e. it is possible to enter '12345678' but not '123456789'. However, when a division to obtain a transfer value is made on such a calculator, the result does not contain eight decimal places, but only seven. Thus, to use the example from the AEC website, (surplus = 992137; transferable papers = 1518178), an 8-digit electronic calculator would display a result of 0.6535050 and not the 0.65350505 quoted. It would be possible to obtain eight significant figures on such a calculator by scaling the calculation, eg 992137 / 151817.8 or 9921370 / 1518178. The transfer value would then be displayed as '6.5350505'. However, there would an additional risk of mistakes being made if calculations were scaled in this way and the increase in precision would be very small.

Taking a practical approach, I would recommend that transfer values should be calculated to 7 decimal places, reflecting the capacity of the commonly available electronic calculators. If the calculation loss is minimised in this way, there is then no need to record decimal fractions of votes for each candidate on the result sheet. The loss that would be incurred in discarding the fractional values when summing the votes for each candidate is very small compared to the calculation loss. This would greatly simplify the presentation of STV-PR result sheets for public elections.

Table 1 Australian Federal Senate Elections 1998
Non-transferable Votes arising on Transfer of Surpluses from First Preferences of Candidates elected at Stage 1

State	Total Vote	Quota	Candidate	Candidate's F P Vote	Surplus	Candidates receiving votes	Exhausted Ballots	Lost by Fraction	LbF as Percentage of Quota
NSW 2	3,755,725	536,533	Heffernan	1,371,578	835,045	35	12	23	0.0043%
NSW 1	3,755,725	536,533	Hutchins	1,446,231	909,698	39	18	25	0.0047%
QLD 3	2,003,710	286,245	Hill	295,903	9,658	15	1	14	0.0049%
VIC 2	2,843,218	406,175	Troeth	1,073,551	667,376	27	9	22	0.0054%
VIC 1	2,843,218	406,175	Conroy	1,148,985	742,810	28	10	24	0.0059%
QLD 1	2,003,710	286,245	McLucas	653,183	366,938	31	15	23	0.0080%
QLD 2	2,003,710	286,245	Parer	568,406	282,161	26	8	24	0.0084%
SA 2	946,816	135,260	Bolkus	301,618	166,358	23	6	13	0.0096%
WA 1	1,063,811	151,974	Ellison	405,617	253,643	26	10	16	0.0105%
WA 2	1,063,811	151,974	Cook	366,874	214,900	33	11	16	0.0105%
SA 1	946,816	135,260	Vanstone	381,361	246,101	27	8	17	0.0126%
ACT	197,035	65,679	Lundy	83,090	17,411	15	4	10	0.0152%
TAS 2	308,377	44,054	Abetz	98,178	54,124	18	18	12	0.0272%
TAS 1	308,377	44,054	O'Brien	121,931	77,877	22	30	14	0.0318%

Table 2 Northern Ireland Assembly Elections 1998
Non-transferable Votes arising on Transfer of Surpluses from First Preferences of Candidates elected at Stage 1

State	Total Vote	Quota	Candidate	Candidate's F P Vote	Surplus	Candidates receiving votes	Non-transferable votes	NTV as Percentage of Quota
East Antrim 2	35,610	5,088	Neeson	5,247	159	11	4.89	0.10%
Belfast East 1	39,593	5,657	Robinson	11,219	5,562	15	6.00	0.11%
South Antrim	43,991	6,285	Wilson	6,691	406	9	10.96	0.17%
Belfast North 2	41,125	5,876	Maginness	6,196	320	15	12.25	0.21%
Upper Bann 1	50,399	7,200	Trimble	12,338	5,138	16	20.30	0.28%
Belfast West 1	41,794	5,971	Adams	9,078	3,107	13	22.10	0.37%
North Antrim	49,697	7,100	Paisley	10,590	3,490	15	28.30	0.40%
East Londonderry	39,564	5,653	Campbell	6,099	446	10	25.44	0.45%
West Tyrone	45,951	6,565	Gibson	8,015	1,450	12	32.29	0.49%
Mid-Ulster 2	49,798	7,115	McGuinness	8,703	1,588	7	45.40	0.64%
Fermanagh & South Tyrone	51,043	7,292	Gallagher	8,135	843	11	50.80	0.70%
Mid-Ulster 1	49,798	7,115	McCrea	10,339	3,224	10	49.60	0.70%
Upper Bann 2	50,399	7,200	Rodgers	9,260	2,060	14	55.36	0.77%
Belfast North 1	41,125	5,876	Dodds	7,476	1,600	15	45.79	0.78%
Belfast West 2	41,794	5,971	Hendron	6,140	169	10	50.80	0.85%
North Down	37,313	5,331	McCartney	8,188	2,857	18	47.55	0.89%
Strangford 1	42,922	6,132	Robinson	9,479	3,347	18	59.80	0.98%
East Antrim 1	35,610	5,088	Beggs	5,764	676	14	49.99	0.98%
Foyle	48,794	6,971	Hume	12,581	5,610	14	69.60	1.00%
Belfast East 2	39,593	5,657	Alderdice	6,144	487	18	58.81	1.04%
Strangford 2	42,922	6,132	Taylor	9,203	3,071	20	73.61	1.20%
South Down	51,353	7,337	McGrady	10,373	3,036	16	90.76	1.24%
Newry & Armagh	54,136	7,734	Mallon	13,582	5,848	13	104.92	1.36%

5 Benefits in local government elections in Scotland

The numbers of electors in the constituencies in both the Australian Federal Senate elections and the Northern Ireland Assembly elections are considerably larger than those likely in the multi-member wards for local government elections in Scotland. It is, therefore, useful to make an assessment of the potential effects of changing the precision of calculation of transfer values from 2dp to 7dp using local data.

For this example I have used Glasgow City Council which has an electorate of 453,552 and 79 councillors. I have examined two possible implementations of STV-PR: nine 8-member wards plus one 7-member ward; and nineteen 4-member wards plus one 3-member ward (Table 3). I have assumed there would be equal numbers of electors per councillor in all wards and a turnout of 50%. I have also assumed that the Labour Party would get 47.58% of the first preference votes (= city-wide average in the 2003 FPTP council elections), that 75% of those first preference votes would be for the party's leading candidate in the ward and that all those papers would be transferable. For the calculation with 7dp I have also truncated the transferred votes to integer values as I recommend above. The results in Table 3 show that the effect of truncating the calculation of transfer values at 2dp could be considerable even in the smaller 4-member wards. The losses when the calculation is truncated at 7dp are negligible.

Table 3 Comparison of Effects of Calculating Transfer Values to 2dp and 7dp

Implementation	*8-member ward*	*4-member ward*
Electorate	45,929	22,964
Valid votes	22,964	11,482
Quota	2,552	2,297
Party FP votes	10,926	5,463
Leading candidate's FP votes	8,194	4,097
Surplus for transfer	5,642	1,800
Transfer value 2dp	0.68	0.43
Transferred votes 2dp	5,571.92	1,761.71
Votes lost by truncation at 2dp	70.08	38.29
Votes lost as percentage of quota	2.75%	1.67%
Transfer value 7dp	0.6885525	0.4393458
Transferred votes 7dp	5641	1,799
Votes lost by truncation at 7dp	1	1
Votes lost as percentage of quota	0.039%	0.044%

The actual loss in transfer value due to truncating the calculation at 2dp compared to truncating at 7dp can vary from 0.0000000 to 0.0099999. The general effect can be assessed by considering only the loss that occurs in the third decimal place. The results in Table 4 have been calculated using the same two example wards as above. The ten potential losses all have equal probabilities of occurrence. The loss due to truncation at 2dp in the 8-member ward will exceed 1% of the quota in six cases out of ten and will exceed 2% in three cases out of ten. Even in the smaller ward, the loss due to this truncation will exceed 1% of the quota in four cases out of ten. These losses are substantial and could be avoided by a simple change to the rules for STV-PR elections.

Table 4 Loss of Votes due to Truncation of Transfer Value before 3dp

Implementation	*8-member ward*		*4-member ward*	
Transferable papers	8,194		4,097	
Loss in transfer value	Votes lost	% of quota	Votes lost	% of quota
0.000	0	0.00%	0	0.00%
0.001	8	0.31%	4	0.17%
0.002	16	0.63%	8	0.35%
0.003	24	0.94%	12	0.52%
0.004	32	1.25%	16	0.70%
0.005	40	1.57%	20	0.87%
0.006	49	1.92%	24	1.04%
0.007	57	2.23%	28	1.22%
0.008	65	2.55%	32	1.39%
0.009	73	2.86%	36	1.57%

6 References

[1] Draft Local Governance (Scotland) Bill: Consultation. October 2003. http://www.scotland.gov.uk/consultations/localgov/dlgsc-00.asp

[2] Election Methods web group. http://electorama.com/modules.php?op=modload&name=News&file=article&sid=3&mode=thread&order=0&thold=0

[3] B A Wichmann (1999). A review of the ERS97 rules. *Voting Matters*, Issue 10, 5-7

[4] R A Newland and F S Britton (1997). *How to conduct an election by the single transferable vote*. Electoral Reform Society, London. 1997 ISBN 0 903291 18 5.

[5] R A Newland and F S Britton (1972). *How to conduct an election by the single transferable vote*. Electoral Reform Society, London. 1973 ISBN 903291 02 9.

[6] Australian Electoral Commission Info Centre. Counting the votes: Senate. http://www.aec.gov.au/_content/what/voting/count_senate.htm

[7] G Newman (2003) Director, Research Section, Australian Electoral Commission. Personal communication.

[8] Commonwealth Electoral Act 1918 Section 273. Scrutiny of votes in Senate elections. http://scaleplus.law.gov.au/html/pasteact/0/57/0/PA003460.htm

[9] Australian Election Archive: Federal Election of 3 October 1998. http://psephos.adam-carr.net/countries/a/australia/index1998.shtml

[10] UK Election Results: Northern Ireland Assembly 1998 — Full STV results. http://www.election.demon.co.uk/niassembly.html

[11] British Statutory Rules and Orders No. 727. 1921.