Divisor Method Proportional Representation in Preference-Ballot Elections

Ross Hyman
rhyman@aflcio.org

Abstract

This paper describes a preference-ballot voting procedure that satisfies proportionality conditions consistent with allocation rules for divisor-method party-list elections such as d'Hondt, Sainte-Laguë, or Huntington-Hill. The procedure generalizes Douglas R. Woodall's Quota Preferential by Quotient procedure, which proportionally assigns candidates to seats in accordance with the d'Hondt divisor method. Variations of the procedure consistent with party-list elections but violating the later-no-harm/help criterion are also presented.

Keywords: Divisor method, Proportional representation, Preference ballot

1 Introduction

Proportional representation in multi-candidate elections is achieved through two different mechanisms in common use today: party-list elections and single transferable vote (STV) preference-ballot elections. This paper describes a preference-ballot voting procedure that is similar to STV but with a proportionality condition satisfied by divisor-method party-list elections instead of the proportionality condition satisfied by STV elections.

The procedure described in this paper can be used in national party-list elections, such as those in Scandinavian countries, to determine the number of seats that each party is awarded to national parliaments. The procedure allows voters to rank parties instead of just voting for one, while retaining the divisor methods that the countries are currently using. The procedure can also be used wherever STV elections can be used.

Party-list proportional representation elections are used in many countries for multi-seat elections to parliaments. In party list elections, the electorate votes for parties not candidates.[1] Seats are awarded to each party in proportion to its vote total, and candidates are elected on the basis of their rankings on party lists that are published before the election.

Perfect proportionality between awarded seats and votes is unachievable. There are two common classes of methods for assigning seats in party-list elections in an approximately proportional way [1, 5, 11]. They are the divisor method (with d'Hondt [2, 3, 10] rounding, or Sainte-Laguë [14] rounding, or Huntington-Hill [8, 9] rounding, etc.) and the largest remainder method (with the Hare [7] quota, or the Droop [4] quota, etc.). Both classes of methods are described in Section 2 of this paper.

In STV [6, 7] elections, which are also used for multi-seat elections to parliaments, voters construct their own ranked list of preferred candidates instead of choosing amongst ready-made party lists. STV uses a quota (Hare, Droop, etc.) to assign seats in an approximately proportional way, and it satisfies a quota-based proportionality condition.

In the October 2003 issue of *Voting matters*, Douglas R. Woodall [18] introduced a preferential voting procedure based upon the divisor method with d'Hondt rounding. The procedure is based on an idea of Olli Salmi [15, 16] to add an elimination procedure to the d'Hondt-

[1] In "open" elections of this type, voters are able to vote for one or more candidates as well, which can reorder the candidates on the party list.

Phragmén method proposed by Lars Edvard Phragmén in 1895 [12, 13].

In this paper, Woodall's procedure is generalized, so that other rounding rules such as Sainte-Laguë or Huntington-Hill can be used to proportionally assign seats in preference-ballot elections. The generalized procedure, like Woodall's d'Hondt procedure and STV, satisfies the later-no-harm/help election criterion. Unlike STV, it satisfies a divisor-method proportionality condition instead of a quota-based proportionality condition.

In addition, Woodall's single-round procedure is modified so that ballot seat values can never decrease.

If each voter is a party loyalist and ranks all of the candidates from their party on their preference ballot in party order, and ranks no other candidates on their ballot, the procedure will *not* in general elect the same number of candidates from each party as the equivalent divisor-method party-list election. (The procedure does still satisfy the divisor-method proportionality condition, which is a less restrictive condition than perfect agreement with party list elections.) Alternative procedures are presented that agree with party list elections when voters vote only party lists, but at the expense of not satisfying the later-no-harm/help election criterion.

Section 2 of this paper introduces largest remainder and divisor methods for assigning seats in party list proportional representation elections. Section 3 introduces the divisor method in priority form, the form needed for preference voting. It also makes the case that Huntington-Hill divisor methods are the only divisor methods that are unbiased between large and small parties. Section 2 and Section 3 can be skipped by those already familiar with divisor methods. In Section 4, the divisor method preference voting procedure satisfying later-no-harm/help is described and demonstrated. Section 5 demonstrates properties of the election procedure including the proportionality condition. Section 6 presents variations of the procedure to reproduce party list elections at the cost of not satisfying later-no-harm/help. Section 7 concludes the paper.

2 Approximately Proportional Methods for Party List Elections

This section introduces largest remainder and divisor methods for proportionally assigning seats in party list elections. In party list elections, seats are awarded to parties in proportion to their vote totals. The numbers of seats, S_i, apportioned to parties are perfectly proportional to votes, V_i, if there is a single quota Q such that $S_i = V_i/Q$ for each party. If votes for each party only came in multiples of the quota, then a party would be assigned one seat for each quota of votes. For example, if 500 voters vote in a party list election for 5 seats and 200 voters choose the Red Party, 200 Voters choose the Green Party, and 100 voters choose the Blue party, dividing each total party vote by 100 assigns 2 seats to the Red Party, 2 seats to the Green Party and 1 seat to the Blue Party.

Since total party votes are generally not integer multiples of the desired quota and seats must be assigned in integer units, perfect proportionally is generally unattainable and rounding is not guaranteed to produce the desired number of total seats. For example, if 500 voters vote in a party list election for 5 seats and 222 voters choose the Red Party, 149 voters choose the Green Party, and 129 voters choose the Blue Party, dividing each total party vote by 100 assigns 2.22 seats to the Red Party, 1.49 seats to the Green Party and 1.29 seats to the Blue Party for a total of five seats. Conventional rounding assigns 2 seats to the Red Party, 1 seat to the Green Party, and 1 seat to the Blue party for a total of only 4 seats.

Approximate proportionality that assigns the desired total number of seats can be achieved through largest remainder or divisor methods. For the largest remainder method, a quota is fixed and the rounding rule is adjusted so that the desired number of candidates is elected. In the above example, 5 seats are assigned if rounding up occurs not at 0.5 but at any number greater than 0.29 but less than or equal to 0.49. This adjusted rounding rule assigns 2 seats to the Red party, 2 seats to the Green party, and 1 seat to the Blue Party, for a total of 5 seats. The largest remainder method is so-called because it is equivalent to rounding up the party seat assignments in decreasing order from the largest fractional remainder to the smallest, until the desired number of seats is assigned. STV is a largest remainder method.[2]

For divisor methods, a rounding rule is fixed and the quota is adjusted so that the desired number of candidates is elected. In the above example, 5 seats are assigned for conventional rounding if party votes were divided not by 100 but by any number greater than 88.8 but less than or equal to 99 ⅓. For example, dividing party votes by 99 instead of 100 assigns 2.242 seats to the Red Party, which rounds down to 2 seats, 1.505 seats to the Green Party, which rounds up to 2 seats, and 1.303 seats to the Blue Party, which rounds down to 1 seat, for a total of 5 seats. This paper presents a procedure for applying divisor methods to preference voting.

In Section 3, the priority formulation of the divisor method, which is needed for preference voting, is introduced.

3 Divisor Methods in Priority Form

This section develops and demonstrates the priority formulation of divisor methods, which will be applied to preference-ballot voting in Section 4. Also, several rounding rules in common use are described, and their bias for small or large parties is shown with an apportionment slide rule.

3.1 An Apportionment Slide Rule

Imagine two sliding rulers, one on top of the other, with logarithmic scaling on each.[3] Let the top ruler be the Votes Ruler and the bottom ruler be the Seats Ruler. For a given positioning of the two rulers, the number of seats awarded to a party is the number of seats on the Seats Ruler directly below the number of votes on the Votes Ruler that a party received. Each positioning of the Votes Ruler with respect to the Seats Ruler corresponds to a different perfect apportionment (before rounding) corresponding to a particular quota.

Different rounding rules can be visualized in the following way. For each integer N, a fixed-rounding mark, $\log(F_{N-1,N})$ is placed between $\log(N-1)$ and $\log(N)$ on the Seats Ruler. For each $\log(N)$, the rounding mark $\log(F_{N-1,N})$ is to its left and the rounding mark $\log(F_{N,N+1})$ is to its right. The segment of the seats ruler between consecutive rounding marks $\log(F_{N-1,N})$ and $\log(F_{N,N+1})$ is the integer seat region for N seats. When the Votes Ruler is positioned over the Seats Ruler so that $\log(V_i)$ is over any part of the N seat region, the i^{th} party is assigned N seats.

3.2 Rounding Rules

Two common rounding rules for party-list proportional representation elections are the Jefferson-d'Hondt rounding rule and the Modified Sainte-Laguë rounding rule. Jefferson-d'Hondt rounding is the same as rounding down. The seat region boundary marks are at $F_{N-1,N} = N$ and the segment from $\log(N)$ to $\log(N+1)$ is the Jefferson-d'Hondt region for N seats.

Modified Sainte-Laguë rounding is conventional rounding, except for $F_{0,1}$. The seat region boundary marks are at $F_{N-1,N} = N - 0.5$ and the segment from $\log(N-0.5)$ to $\log(N+0.5)$ is the Sainte-Laguë region for N seats. Modified Sainte-Laguë, sets $F_{0,1} = 0.7$, instead of the unmodified 0.5, making it harder for a small party to gain a seat. We will see below that all values of $F_{0,1} \leq F_{1,2}/2 = 0.75$ are admissible for preference voting. Because the Sainte-Laguë rounding marks are closer to the rightmost integer than the leftmost integer on a logarithmic scale, more seats will be rounded down than rounded up.

On a logarithmic scale, the distance between consecutive integers decreases as the integers increase. Because of this, when the number of seats apportioned to a party is rounded to an integer, the shift away from perfect proportionality is greater for a party with a small number of votes than it is for a party with a large number of votes. For this reason, a rounding rule that rounds down more than it rounds up (such as d'Hondt[4] or SainteLaguë) is biased against

[2] For Meek's method and for some other STV systems, the quota is recalculated when ballots become inactive.

[3] On a logarithmic scale the distance between two numbers is proportional to their ratio.

[4] d'Hondt rounding's bias in favor of large parties is often counted as a point in its favor since

small parties compared to large parties and a rounding rule that rounds up more than it rounds down is biased in favour of small parties compared to large parties.

The only rounding rule that isn't systematically biased on a logarithmic scale is one with rounding marks placed exactly between the integers on such a scale.[5] The Huntington-Hill rounding rule, which is used in the United States to apportion the seats of the House of Representatives to the states, is defined in this way. The Huntington-Hill rounding mark between $\log(N-1)$ and $\log(N)$ is half way between them:[6] that is,

$$\log(F_{N-1,\,N}) = \tfrac{1}{2}\,(\log(N-1) + \log(N)),$$

so

$$F_{N-1,\,N} = \sqrt{(N(N-1))},$$

the geometric mean. This assigns the integer region for N seats to the region of the slide rule closest to $\log(N)$.

Without modification, Huntington-Hill awards a seat to any candidate getting just one first choice vote, since $F_{0,\,1} = 0$. An increased $F_{0,\,1}$ above zero makes Huntington-Hill viable for proportional representation elections. We will see below that all values of $F_{0,\,1} \le F_{1,\,2}/2 = \sqrt{½}$ are permissible for preference voting. Modified Huntington-Hill with $F_{0,\,1} = \sqrt{½}$ agrees with Jefferson-d'Hondt when all parties receive 2 or fewer seats. Since $\sqrt{½}$ is approximately 0.7, and $\sqrt{(N(N-1))}$ is approximately $N-0.5$ for large N, modified Huntington-Hill with $F_{0,\,1} = \sqrt{½}$ is similar to modified Sainte-Laguë.[7]

3.3 Priority/Load Formalism

If the Votes Ruler is positioned over the Seats Ruler such that $\log(V)$ votes on the Votes Ruler is positioned directly over $\log(S)$ seats on Seats Ruler then the quota is V/S and the fraction of seats that each ballot accounts for is S/V. Due to the magic of logarithms, every $\log(V)$ and every $\log(S)$ that are positioned directly over each other on the two rulers have the same V/S ratio for a given positioning of the two rulers.

The slide rule can systematically assign seats to parties by placing the Votes Ruler to the left of the Seats Ruler and moving it to the right, which decreases the quota and increases the seat fraction per ballot. Each time the vote mark for the i^{th} party crosses a rounding mark, the i^{th} party acquires an additional seat.

When $\log(V_i)$ on the Votes Ruler is directly over $\log(F_{N-1,\,N})$ on the Seats Ruler, the i^{th} party crosses from the $N-1$ seat region to the N seat region and acquires its N^{th} seat. The quota for when this occurs is $V_i/F_{N-1,\,N}$. This is the *priority* or *quotient* for the i^{th} party to have N seats. The inverse priority, $F_{N-1,\,N}/V_i$, which without rounding is the average number of seats per ballot, is the *load* [16] for the i^{th} party to have N seats.[8] One calculates priority quotients or loads for parties to acquire seats and assigns seats to the parties in order from highest priority to lowest, or lowest load to highest, stopping when the appropriate number of seats has been reached. For party-list elections the priority formalism is commonly used. Phragmén invoked the load formalism for his preference-ballot procedure [12, 13, 16]. A priority tends to be a large number divided by a small number while a load tends to be a small number divided by a large number. We will find that the load

---

it discourages party splits and encourages party mergers. Only d'Hondt rounding guarantees that a majority of voters will be awarded a majority of seats. In Sainte-Laguë and Huntington-Hill rounding, a majority could have its seats rounded down while a minority has its seats rounded up, resulting in a majority rule violation.

[5] Balinsky and Young argue that Huntington-Hill is more biased than Sainte-Lague. However, they did not use a logarithmic scale in defining their bias criteria.

[6] This is why Huntington called his method "Equal Proportions."

[7] The choice $F_{0,\,1} = \sqrt{½}$ is also motivated by allowing inverse integer seat regions, $1/N$, between 0 and 1, which are the mirror images, on a logarithmic scale, of the integer seat regions between 1 and infinity. For these additional regions, $F_{1/(N+1),\,1/N} = \sqrt{((1/(N+1))1/N)}$, and in particular $F_{1/2,\,1} = \sqrt{½}$ is the rounding mark between the ½ seat region and the 1 seat region. Since seats can only be assigned in integer units, parties that would receive fractional seats are excluded.

[8] Phragmén uses the term belastnig, which Olli Salmi translates as load, for what we call seat value [13].

formalism is more natural for preference voting when presented in the abstract but that priorities have the advantage over loads, when concrete examples are presented, of being easier to calculate and compare magnitudes by hand.

For an example of the priority (load) formulation, consider the party-list election for 5 seats in which 222 voters choose the Red Party, 149 voters choose the Green Party, and 129 voters choose the Blue Party. For simplicity, Jefferson-d'Hondt rounding is used. The five highest priorities (lowest loads) are the Red Party's priority for one seat, 222/1=222 (1/222 = 0.0045), the Green Party's priority for one seat, 149/1=149 (1/149 = 0.0067) the Blue Party's priority for one seat, 129/1=129 (1/129 = 0.0078), the Red Party's priority for two seats, 222/2 = 111 (2/222 = 0.009), and the Green Party's priority for two seats, 129/2 = 64.5 (2/129 = 0.016). All other priorities are lower than these. The Red Party is apportioned two seats, the Green Party is apportioned two seats, and the Blue Party is apportioned one seat.

The priority/load formulation of fixed-rounding is generalized to preference-ballot elections in Section 4.

4 Proportional Preference-Ballot Voting by the Divisor Method

This section develops a divisor method for electing candidates in preference-ballot elections. The method is a generalization of the priority/load formalism for divisor method party-list elections described above, and of Woodall's Quota Preferential by Quotient procedure for d'Hondt rounding. The procedure described in this section satisfies the later-no-harm/help criteria but is not guaranteed to agree with the results of a party-list election if each voter votes a party list.

The single-round, d'Hondt version of the election procedure described in this section differs from Woodall's single-round procedure slightly in that seat values are guaranteed never to decrease. The multi-round, d'Hondt version of the election procedure is identical to Woodall's multi-round procedure.

The procedure can be visualized by imagining a Votes Ruler, as before, but now also many Seats Rulers, one for each ballot. Since each ballot counts for one vote, the Votes Ruler has a mark at $V = 1$ and nowhere else. The values of the seat regions on the Seats Rulers can be any number, not just integers, and their values and rounding marks can be different for each ballot and are determined as the election count proceeds.

We begin with a series of definitions.

Elected, hopeful, and excluded candidates

Following Woodall, each candidate is in one of three states, designated as elected, hopeful and excluded. At the start of the first stage, every candidate is hopeful. As the count proceeds hopeful candidates are reclassified as elected or excluded.

Active and inactive ballots

Following Woodall, a ballot is active when it ranks at least one hopeful candidate. It is inactive when it ranks no hopeful candidate.

The seat value of a ballot

Following Woodall, each ballot is assigned a seat value that corresponds to the fractional number of candidates that the ballot can be said to have elected. The seat value of ballots cannot decrease (the exception is in the multi-round version of the procedure, when the count is restarted and all seat values are reset to zero). The sum of seat values over all ballots is the current number of elected candidates.

The candidate election load

In the priority/load formalism for divisor-method party-list elections, the Votes Ruler is shifted to the right and a new seat is acquired by the i^{th} party each time the Vote mark for the i^{th} party crosses the next rounding mark on the seat ruler. We will perform the same procedure for preference voting, except that seat regions and rounding marks are not fixed beforehand. Instead, as we shift the Votes Ruler to the right, a trial rounding mark on each ballot's Seat Ruler directly follows underneath the $V = 1$ mark on the Votes ruler. At any particular positioning of the Votes Ruler with respect to the Seat Rulers, the value of the trial rounding mark, f, and the ballot's seat value s, determine the seat value of the trial seat region to its right according to the formula $s' = g(s, f)$, where $g(s, f)$ is a function that depends on which divisor method rounding rule is being used. At some point, as the Votes Ruler and trial rounding mark moves to the right, the sum of the trial seat values, s', for all of the ballots with candidate c as the topmost hopeful candidate will equal the sum of the current seat values for those ballots + 1. The value of the trial rounding mark at that point is f_c, the load to elect candidate c. The priority to elect candidate c is $p_c = 1/f_c$. These definitions are consistent with the party-list definitions for the load and priority of a candidate to be elected. The load to elect candidate c satisfies

$$\sum_{cballots}\left(g\left(s, f_c\right) - s\right) = 1,$$

where the sum is taken over all ballots with topmost active hopeful candidate c. In the above and all subsequent ballot sums, the ballot index for each seat value has been suppressed. It is important to keep in mind that seat values are for ballots and can be different for each ballot in the sum, while a load is for a candidate (or a group of candidates, as we will see below) and is a constant in the ballot sum.

4.1 Properties of $g(s, f)$

A ballot with seat value s has its seat value increased to $s' = g(s, f)$ when its topmost hopeful candidate is elected with load f. The function $g(s,f)$ must satisfy the following conditions:

a) $g(s,f) \geq f$ for $f > s$,

b) $g(K-1, F_{K-1,K}) = K$,

c) $ag(s,f) = g(as, af)$,

d) $g(s,f) = s$ when $s \geq f$,

e) $g(s,f)$ is monotonically decreasing in s for $s \leq f$, that is $g(s,f)$ does not increase when s increases with f held fixed, and

f) $g(s,f)$ is strictly increasing in f for $f \geq s$ and s held fixed.

Condition a) guarantees that a ballot's seat value cannot decrease. Conditions b) and c) together guarantee that $g((K-1)/V, F_{K-1,K}/V) = K/V$ for any V, which is required for consistency with divisor method rounding rules. Condition d) allows ballot sums to include ballots with seat values larger than the load. Condition e) is required to guarantee that electable candidates remain electable. Condition f) guarantees that there is one and only one f_c for each candidate c. Condition f) is not an independent condition. It is a consequence of condition a), condition c), and condition e) which together guarantee that $\partial g/\partial f \geq 1$ for $f \geq s$.

4.2 Rounding Rules

All of the following rounding rules have $g(s, f)$ functions that satisfy the above conditions. They are determined by inverting the rounding mark formulas $f = f(s, s')$. For d'Hondt, $g(s, f) = \max(s, f)$. For unmodified Sainte-Laguë $g(s, f) = \max(s, 2f - s)$. For unmodified Huntington-Hill $g(s, f) = \max(s, f^2/s)$. Modified Huntington-Hill and modified Sainte-Laguë rounding have $F_{0,1}$ above their unmodified values. Condition c) guarantees that $g(s, f) = f\,h(s/f)$ where $h(x)$ is a function of one variable. Applying condition b) we have $h(0) = 1/F_{0,1}$ and $h(1/F_{1,2}) = 2/F_{1,2}$. There are many ways to extrapolate $h(x)$ between these points that satisfy the rounding rule conditions. A linear extrapolation leads to

$$h(x) = 1/F_{0,1} + (2 - F_{1,2}/F_{0,1})x$$

for $x \leq 1/F_{1,2}$, with unmodified $h(x)$ for $x \geq 1/F_{1,2}$. Hence, for $f \geq F_{1,2}$,

$$g(s, f) = \frac{f}{F_{0,1}} - \left(F_{1,2} - 2F_{0,1}\right)\frac{s}{F_{0,1}},$$

with unmodified $g(s,f)$ for $f \leq F_{1,2}s$. Condition e) requires that $F_{0,1} \leq F_{1,2}/2$. For modified Huntington-Hill, in which $F_{0,1} = F_{1,2}/2 = (\sqrt{2})/2$, $g(s,f) = (\sqrt{2})f$ for $f \geq (\sqrt{2})s$, and $g(s,f) = \max(s, f^2/s)$ otherwise. For modified Sainte-Laguë with $F_{0,1} = F_{1,2}/2 = 0.75$, $g(s, f) = 4f/3$ for $f \geq 1.5s$ and $g(s, f) = \max(s, 2f - s)$ otherwise. For modified Sainte-Laguë with $F_{0,1} = 0.7$, $g(s, f) = (10f - s)/7$ for $f \geq 1.5s$ and $g(s, f) = \max(s, 2f - s)$ otherwise.

4.3 The Electability Load

The fact that candidate c has the lowest election load does not mean that candidate c should necessarily be elected. It could be that all the voters who voted for candidates other than c command enough votes to fill all of the remaining seats with candidates other than c, and at lower loads than the load to elect c, if only they had voted more strategically. The lowest possible load to fill the remaining seats with non-c candidates, f_{notc}, satisfies

$$\sum_{notcballots}\left(g\left(s, f_{notc}\right)-s\right)=R \ ,$$

where the sum is over all active ballots in which c is not the topmost active hopeful candidate, and R is the remaining number of seats to be filled.[9] Hopeful candidate c is electable when $f_c < f_{notc}$

It is not necessary to calculate f_{notc} to determine whether c is electable. The electability load, f_{elect}, satisfying

$$\sum_{active}\left(g\left(s, f_{elect}\right)-s\right)=R+1 \ ,$$

where the sum is over all active ballots, is always between f_c and f_{notc} and therefore can be used as an alternative electability criteria for c. Hopeful candidate c is electable when $f_c < f_{elect}$

Proof: If f_{elect} were less than f_c and f_{notc} then the sum on the LHS would be less than $R + 1$. If f_{elect} were greater than f_c and f_{notc} then the sum on the LHS would be greater than $R + 1$.

One consequence of this fact is that if f_c is the lowest election load and $f_{notc} \le f_c$ so that c is not electable, then no hopeful candidate is electable. The electability load serves a similar purpose to the quota in STV elections, of determining whether a hopeful candidate is electable. In divisor methods, at any stage, the quota is the range of values with a maximum equal to the election priority of the electable candidate with lowest election priority and with a minimum that is just greater than the election priority for the unelectable candidate with highest election priority. The electability priority $Q = 1/f_{elect}$ always falls in this range, so it is a valid quota. This is the generalization of the quota, Q, used in Woodall's paper.

4.4 Explicit Load Formulas

The election loads and electability load are determined from implicit formulas of the form

$$\sum_{s}\left(g\left(s, f\right)-s\right)-M=0 \ ,$$

differing only in which ballots are summed and the value of M. Since $g(s, f)$ is strictly increasing in f for $f > s$, the election load and electability load equations have unique solutions. Inverting load equations is complicated by the fact that $g(s, f)$ is piecewise continuous, with different formulas when s is less than or greater than f and for modified rules, when s is less than or greater than $f/F_{1,2}$ An iterative method to find f in such equations is as follows.

Step 1. All ballots that are included in the sum are placed into groups in increasing seat value order: $s_1 < s_2 < s_3$ etc. The number of ballots in the k^{th} group is V_k.

Step 2. Calculate the next iteration of f from one of the following formulas. For the first iteration include all ballots in the following sum and for the previous value of f, choose infinity. For later iterations, include only those ballots with seat values less than the previous value of f.

For unmodified rules and modified rules in which every seat value that is less than f is larger than $f/F_{1,2}$ use $f =$

$$\frac{F_{0,1}M+\sum_{k=1}^{r}V_k s_k}{\sum_{m=1}^{r}V_m}$$

for d'Hondt ($F_{0,1} = 1$) and unmodified Sainte-Laguë ($F_{0,1} = 0.5$), and $f =$

$$\sqrt{\frac{M+\sum_{k=1}^{r}V_k s_k}{\sum_{m=1}^{r}\frac{V_m}{s_m}}}$$

for unmodified Huntington-Hill.

For modified rules in which every seat value that is less than f is larger than $f/F_{1,2}$ use $f =$

[9] This distribution of non-c candidates is not necessarily attainable since it requires each voter to split their ballot into R equal pieces and vote for one of R non-c candidates on each piece, each split ballot counting for $1/R$ of a vote. However, the attainability of the distribution is not as important as the fact that the electability criterion leads to the desired proportionality condition, as will be proved below.

$$\frac{F_{0,1}M + \left(F_{1,2} - F_{0,1}\right)\sum_{k=1}^{r} V_k s_k}{\sum_{m=1}^{r} V_m}.$$

Otherwise use f =

$$\frac{\sqrt{2\left(\sum_{k=1}^{p} V_k\right)^2 + 4\left(\sum_{k=p+1}^{r} \frac{V_k}{s_k}\right)\left(M + \sum_{k=1}^{r} V_k s_k\right)} - \sqrt{2}\sum_{k=1}^{p} V_k}{2\sum_{k=p+1}^{r} \frac{V_k}{s_k}}$$

for modified Huntington-Hill and f =

$$\frac{F_{0,1}M + \left(1.5 - 2F_{0,1}\right)\sum_{k=1}^{p} V_k s_k + F_{0,1}\sum_{k=p+1}^{r} V_k s_k + F_{0,1}\sum_{k=1}^{r} V_k s_k}{\sum_{k=1}^{p} V_k + 2F_{0,1}\sum_{k=p+1}^{r} V_k}$$

for modified Sainte-Laguë. In the above expressions, s_p is the largest seat value that is less than the current value of $f/F_{1,2}$ and s_r is the largest seat value less than the current value of f.

Step 3. Repeat Step 2 until an f has been found such that s_p and s_r are unchanged. That value of f is the correct load.

In the following election procedure Woodall's d'Hondt single-round procedure [18] is generalized to other divisor methods.

4.5 Election Procedure 1

The following is a single-round election procedure for N seats that satisfies both later-no-harm/ help and a divisor-method proportionality condition.

Step 1. At the start of the first stage every candidate is hopeful and the seat value of every ballot is zero. The remaining number of seats to be filled, R, is set to N, the total number of seats to be filled.

Step 2. The election load f_c for each hopeful candidate c that is the topmost hopeful candidate on at least one ballot is determined from

$$\sum_{cballots}\left(g\left(s, f_c\right) - s\right) = 1,$$

where the sum is taken over all ballots where c is the topmost hopeful candidate and the electability load is determined from

$$\sum_{active}\left(g\left(s, f_{elect}\right) - s\right) = R + 1,$$

where the sum is taken over all active ballots. If at least one hopeful candidate is electable, that is, $f_c < f_{elect}$, go to step 3a. If no candidate is electable, go to step 3b.

Step 3a. The electable candidate with the lowest election load is elected. (If the total number of elected candidates is N, the count can be ended since no more candidates will be elected). R is reduced by 1. If candidate c is elected, the seat value for each ballot with seat value s that contributed to electing c is increased to $g(s, f_c)$. Proceed to Step 2.

Step 3b. Exclude the candidate with the largest election load amongst those that are the topmost hopeful candidate on at least one ballot. Also exclude all hopeful candidates that do not appear as the topmost hopeful candidate on any ballot. (If the total number of elected plus hopeful candidates is N then all of the hopeful candidates can be elected and the count ended since they are all guaranteed to be elected.) Proceed to Step 2.

The single-round procedure with d'Hondt rounding differs from Woodall's single-round procedure in the calculation of loads/priorities, so that seat values cannot decrease. This is demonstrated with Election 1 from Woodall's paper. Loads rather than priorities will be presented to be consistent with the formulism used throughout this paper. However, all loads will be presented as the inverse of priorities for easy comparison with Woodall's examples and the priority formalism.

Election 1 (3 seats, d'Hondt)

16	AB
12	B
12	C
12	D
8	EB

Stage 1: The election and electability loads are $f_A = 1/16$, $f_B = 1/12$, $f_C = 1/12$, $f_D = 1/12$, $f_E = 1/8$, and $f_{elect} = 4/60 = 1/15$. The lowest election load is f_A. It is lower than f_{elect}. Candidate A is elected. Each of the seat values for the 16 ballots ranking candidate A first is increased from zero to 1/16. Stage 2: Candidate B's election load is decreased to $f_B = 2/(12+16) = 1/14$. The

other loads are unchanged. No candidate's election load is lower than the electability load so candidate E, who has the highest election load, is excluded. Candidate B's election load is again decreased, this time to 1/20. This is calculated by only including the ballots ranking B first (not 2/36= 1/18, as that incorrectly includes the 16 ballots with seat values of 1/16 which do not contribute to B's load since 1/16 is greater than 1/18). Candidate B is elected. Each of the seat values for the 20 ballots ranking B first is increased from zero to 1/20. Stage 3: f_C = 1/12, f_D = 1/12, f_{elect} = 2/24 = 1/12. Neither of the remaining candidates is electable. One must be excluded and the other is elected.

In the next section, properties that Woodall proved for d'Hondt rounding are proved for the general case. The section culminates in a divisor method proportionality condition.

5. Properties of the Election Procedure

The goal of this section is to prove that the election procedure described in the previous section satisfies a divisor method proportionality condition. This is done through a serious of steps following the logic Woodall used to demonstrate d'Hondt proportionality (which turns out to be the same as Droop proportionality). The first two proofs together combine to prove that an electable candidate remains electable. The next two proofs together combine to prove that all electable candidates will eventually be elected. From there the proportionality condition is proved by considering the worst case scenario in which a candidate is electable.

Election loads of hopeful candidates cannot increase

Electing and excluding candidates other than hopeful candidate c can increase but can never decrease the number of ballots in which c is the topmost hopeful candidate. More ballots means that more can contribute to

$$\sum_{cballots}\left(g(s,f_c)-s\right),$$

which from properties a), d), and f) cannot increase the f_c required to bring the sum to one.

The electability load cannot decrease

Excluding candidates does not change the seat values of ballots. It can cause some ballots to become inactive which can only increase f_{elect}. When a candidate is elected, the sum of seat values of all ballots increases by one and the remaining number of seats, R, decreases by one. Moving the seat value sum to the RHS in the defining equation for the electability load we have a new RHS that is unchanged after an election, provided no new ballots have become inactive:

$$\sum_{active} g(s,f_{elect}) = 1 + R + \sum_{active} s\ .$$

When no ballots become inactive the electability load cannot decease after an election of candidate c with load $f_c < f_{elect}$, since $g(g(s,f_c),f_{elect}) \le g(s,f_{elect})$ and $g(s,f)$ is monotonically increasing in f. The requirement $g(g(s,f_c),f_{elect}) \le g(s,f_{elect})$ is true for the case $g(s,f_c) \le f_{elect}$ since $g(s,f_c) \ge s$ and $g(s,f_{elect})$ is monotonically decreasing in s for $s \le f_{elect}$. It is true for the case $g(s,f_c) \ge f_{elect}$ since in that case $g(g(s,f_c),f_{elect}) = g(s,f_c) \le g(s,f_{elect})$ and $g(s,f)$ is monotonically increasing in f.

In addition to the above, at each stage some ballots can become inactive. Fewer active ballots means that fewer will contribute to

$$\sum_{active}\left(g(s,f_{elect})-s\right),$$

which, from properties a), d), and f), cannot decrease the f_{elect} needed to bring the sum to R + 1.

If c is electable, it will remain electable

Since a candidate's election load can only decrease and f_{elect} can only increase, if c is electable at one stage it will remain electable at later stages.

At any stage, at most R hopeful candidates are electable

There is only a possibility of more than R electable candidates if there are more than R hopeful candidates. If there are more that R hopeful candidates, let f_{large} be the $R+1^{\text{th}}$ smallest load of the more than R hopeful candidates. Call the hopeful candidates with rounding marks less than or equal to f_{large} the smaller candidates. It must be the case that

$$\sum_{active}\left(g\left(s, f_{large}\right) - s\right) = \sum_{hopefuls}\ \sum_{cballots}\left(g(s, f_{large}) - s\right)$$
$$\geq \sum_{smalls}\ \sum_{cballots}\left(g(s, f_{large}) - s\right) \geq \sum_{smalls} 1 \geq R + 1,$$

which makes $f_{large} \geq f_{elect}$ and therefore all candidates with election loads greater than or equal to f_{large} are unelectable, so at most R are electable.

If there are R remaining hopeful candidates then at least one is electable

Let $f_{smallest}$ be the smallest load of the R hopeful candidates. It must be the case that

$$\sum_{active}\left(g\left(s, f_{smallest}\right) - s\right)$$
$$= \sum_{hopefuls}\ \sum_{cballots}\left(g(s, f_{smallest}) - s\right) \leq \sum_{hopefuls} 1 = R,$$

which makes $f_{smallest} < f_{elect}$, so the hopeful candidate with the smallest election load is electable when there are R hopeful candidates remaining. (This argument can also be used to show that if there are R+1 remaining hopeful candidates then at least one is electable or all hopeful candidates are tied with election loads equal to the electability load.)

An electable candidate is guaranteed to be elected

Removing the premature stopping condition in parenthesis in Step 3a (that the procedure stops when N candidates are elected) can have no effect since after N candidates are elected, no additional candidates can be elected since none of the remaining hopeful candidates will be electable. Likewise, removing the premature stopping condition in parenthesis in Step 3b (that the procedure elects the remaining hopeful candidates and stops when the total number of elected plus hopeful candidates equals N) also can have no effect since after the total number of elected plus hopeful candidates equals N, the procedure will still elect all of the remaining hopeful candidates, since at least one will always be electable. Since the premature stopping conditions can be removed without changing which candidates are elected, and without the premature stopping conditions the election procedure ends when all candidates are either elected or excluded, and electable candidates cannot be excluded, an electable candidate is guaranteed to be elected.

Proportionality condition

The conditions above insure that the count cannot end before all electable candidates are elected. Therefore demonstrating that a candidate is electable is equivalent to proving that it will be elected. From this we can prove the following proportionality condition.

If there are N seats to be filled and V_T total valid ballots and V ballots all rank the same L candidates higher than all other candidates then at least $K \leq L$ of those candidates will be elected if

$$\frac{V}{V_T} > \frac{\widetilde{F}_{K-1,\,K}}{(N - K + 1)F_{0,\,1} + \widetilde{F}_{K-1,\,K}} \ ,$$

where $\widetilde{F}_{K-1,\,K} = F_{K-1,\,K}$ for d'Hondt, unmodified Sainte-Laguë, and unmodified Huntington-Hill and $\widetilde{F}_{K-1,\,K} = K - 1 + F_{0,\,1}$ for modified Sainte-Laguë for $0.75 \geq F_{0,\,1} \geq 0.5$ and modified Huntington-Hill for $\sqrt{½} \geq F_{0,\,1} \geq 1/2$.

Proof: Consider the worst case scenario to elect K candidates, which is that the K candidates appear only on the V ballots and not on any others. Assume that $K - 1$ of the candidates have already been elected. The load to elect the K^{th} candidate is determined by

$$\sum_{vballots} g(s, f) = K \ ,$$

where the sum is over the V ballots and the seat values satisfy

$$\sum_{vballots} s = K - 1 \ .$$

The maximum value of f is found by minimizing

$$\sum_{vballots} g(s, f)$$

with respect to s with the above seat value constraint and then increasing f until

$$\sum_{vballots} g(s, f) = K \ .$$

The minimum for convex functions is $s = (K - 1)/V$ for each seat value in the sum, from which

$$K = Vg\left(\frac{K-1}{V}, f\right) = g\left(K - 1, Vf\right),$$

where $ag(s, f) = g(as, af)$ has been used. The solution, using

$$K = g\left(K-1, F_{K-1,\,K}\right)$$

is

$$f = \frac{F_{K-1,\,K}}{V}\,.$$

This is the maximum load to elect K candidates for d'Hondt, unmodified Sainte-Laguë and unmodified Huntington-Hill which all have a $g(s, f)$ that is convex in s. Modified Sainte-Laguë for $0.75 \ge F_{0,1} \ge 0.5$ and modified Huntington-Hill for $\sqrt{½} \ge F_{0,1} \ge 1/2$ have non-convex $g(s, f)$ in which a straight line connecting $g(0, f)$ to $g(f, f)$ is lower than $g(s, f)$ at every point along the line.[10] Therefore, for these rounding rules,

$$\sum_{vballots} g(s, f)$$

is minimized by V_2 ballots with $s = 0$ and V_1 ballots with $s = f = (K-1)/V_1$, so that

$$K = V_2 g\left(0, \frac{K-1}{V_1}\right) + V_1 g\left(\frac{K-1}{V_1}, \frac{K-1}{V_1}\right)$$
$$= \frac{V_2}{F_{0,1}}\frac{K-1}{V_1} + K - 1,$$

where $g(0, f) = f/F_{0,1}$ (required by properties b and c), and $g(x, x) = x$ (property d) have been used. The solution is $f = (K-1)/V_1 = F_{0,1}/V_2$. Solving for f in terms of $V = V_1 + V_2$ produces $f = (K - 1 + F_{0,1})/V$.

The load to elect $N - K + 1$ candidates other than the K^{th} candidate so that the K^{th} candidate cannot be elected satisfies

$$\sum_{notvballots}\left(g\left(s, f_{notv}\right) - s\right) = N - K + 1\,.$$

These candidates occur as topmost hopeful candidates only on the $V_T - V$ ballots. The lowest possible load is found when the seats values are as small as possible, which is s = 0. The minimum load to elect $N - K + 1$ candidates is determined by

$$N - K + 1 = \left(V_T - V\right)g(0, f_{notv}) = \frac{\left(V_T - V\right)f_{notv}}{F_{0,1}}$$

The solution is

$$f_{notv} = \frac{\left(N - K + 1\right)F_{0,1}}{V_T - V}\,.$$

[10] In classical thermodynamics this is called "Maxwell's construction" for minimizing non-convex free energy functions.

The K^{th} candidate is electable if $f < f_{notv}$. When rearranged, this is the proportionality condition given above.

A consequence of the proportionality condition is that only d'Hondt rounding guarantees that a majority of voters will be awarded a majority of seats. The d'Hondt proportionality condition guarantees that if the number of seats is $2m + 1$ and there is a voting block that commands more than half of the ballots, then at least $m + 1$ of the seats will be awarded to that voting block. This is not guaranteed for other rounding rules. This is true for party-list elections and preference-ballot elections. The other rounding rules give greater weight to the first ranked candidate, so for majority rule to be violated the majority must rank their candidates mostly the same while the minority distributes the first ranked position more equally amongst their preferred candidates. For party-list elections the number of parties is typically much less than the number of seats, so the extreme circumstances required for majority rule to be violated are much less likely to occur.

6 Variations of the Procedure

In this section, variations of the election procedure are presented that satisfy different voting system criteria.

6.1 Election Procedure 2: A Single-Round Procedure Agreeing with Party-List Elections

The election procedure described in Section 4 can fail to reproduce the result of a party-list election when each voter votes a party-list. This problem exists for STV elections too and is caused by incorrectly excluding candidates because of an artificially small electability load (an artificially large quota) caused by the presence of ballots that become inactive later in the count. Election 2 is an example of this failure.

Election 2 (2 seats, d'Hondt)

90 A1 A2 …
44 B1 B2 …
43 C1 C2 …
41 D1 D2 …
36 E1 E2 …
20 F1 F2 …

The d'Hondt divisor method applied to the equivalent party-list election elects A1 and A2. The preference-ballot procedure with permanent exclusions excludes A2 and elects A1 and B1. However, candidates A1 and A2 are elected if the preference-ballot election procedure is altered so that all excluded candidates are recalled to hopeful status every time a candidate is elected and the premature stopping condition in Step 3b is removed. The altered procedure proceeds as follows. Stage 1: f_{A1} = 1/90, f_{B1} = 1/44, f_{C1} = 1/43, f_{D1} = 1/41, f_{E1} = 1/36, f_{F1} = 1/20, f_{elect} = 3/274 = 1/91.33. The election loads of other candidates are not calculated as they are not the topmost hopeful candidates on any ballots. No candidate's election load is less than the electability load, so candidate F1, with the largest election load, is excluded. Also excluded are all candidates that are not the topmost hopeful candidate on any ballot, including A2. The electability load increases to f_{elect} = 3/254 = 1/84.67. The election loads of the remaining hopeful candidates are unchanged. Candidate A1 is elected. The 90 ballots ranking A1 first are assigned seat value 1/90. Stage 2: All excluded candidates are recalled to hopeful status. The loads are f_{A2} = 2/90 =1/45, f_{B1} = 1/44, f_{C1} = 1/43, f_{D1} = 1/41, f_{E1} = 1/36, f_{F1} = 1/20, f_{elect} = 2/184 = 1/92 (not 1/91.33 as the 90 ballots with seat value 1/90 don't contribute). The election loads of other candidates are not calculated as they are not the topmost hopeful candidates on any ballots. No candidate's election load is less than the electability load, so candidate F1, with the largest election load, is excluded. Also excluded are all candidates that are not the topmost hopeful candidate on any ballot. The electability load is increased to f_{elect} = 1/84.67. No candidate's election load is less than the electability load, so candidate E1 is excluded. The procedure continues with, D1 and C1 successively excluded, at which point the electability load is increased to f_{elect} = 3/134 = 44.67, and candidate A2 is elected.

Temporarily rather than permanently excluding candidates can violate later-no-harm/help. Elections 3 and 4 are examples of this violation.

Election 3 (2 seats, d'Hondt)

18 A
15 AB
24 C
23 D
20 BA

Stage 1: f_A = 1/33, f_B = 1/20, f_C = 1/24, f_D = 1/23, f_{elect} = 1/33.33. Candidate B is excluded. Candidate A's load decreases to f_A = 1/53. Other loads are unchanged. Candidate A is elected. Each ballot electing candidate A is assigned a seat value of 1/53. Stage 2: Candidate B is recalled to hopeful status. Candidate B's election load decreases to f_B = (35/53 +1)/35 = 1/21.08. Candidate C has the lowest election load and is eventually elected.

Had the 20 voters ranking candidate B before candidate A been aware that candidate A would be elected without their help, these voters could have left candidate A off their ballots to increase the chance of their favoured candidate, B, winning the second seat. This is demonstrated in Election 4.

Election 4 (2 seats, d'Hondt)

18 A
15 AB
24 C
23 D
20 B

Stage 1: f_A = 1/33, f_B = 1/20, f_C = 1/24, f_D = 1/23, f_{elect} = 1/33.33. Candidate B is excluded. The election load is increased to f_{elect} = 3/80 = 1/26.67. Other loads are unchanged. Candidate A is elected. Each ballot electing candidate A is assigned a seat value of 1/33. Stage 2: Candidate B is recalled to hopeful status. Candidate B's election load decreases to f_B = (15/33 +1)/35 = 1/24.06. Candidate B has the lowest election load and is eventually elected.

This is a violation of later-no-harm/help since candidate B's election was achieved by removing candidates ranked below B on ballots. The example demonstrates that the procedure encourages free riding, which is the same tactical voting procedure encouraged by all proportional multi-seat preference voting systems, including those that satisfy later-no-harm/help. It is advantageous for some voters to be free riders by not ranking very popular candidates, so that more of their vote will count for their favoured unpopular candidates. But the temptation to be a free rider is tempered by the knowledge that if all voters acted in that way, the popular candidates would lose. It is unclear how the incentive to be a free rider under the procedure violating later-no-harm/help compares to the incentive under the procedure that satisfies later-no-harm/help.

6.2 Election Procedure 3: A Multi-Round Procedure Satisfying Later-No-Harm/ Help and Providing No Benefit to Woodall Free-Riding

Woodall proposed a multi-round version of his election procedure in which the election is rerun after each exclusion, which has the effect of reassigning seat values on ballots to what they would be if the excluded candidate had never run. The multi-round procedure prevents any benefit from what Markus Schulze [17] refers to as Woodall free riding, in which a voter ranks an unpopular candidate she is confident will be excluded above a popular candidate she is confident will be elected so that more of her vote will be counted for lower ranked candidates. It is easily generalized to Election Procedure 1, by recalling all elected candidates to hopeful status, setting all seat values to zero, and setting $R = N$ at the end of Step 3b in Procedure 1 before proceeding to Step 2. However, this procedure, like Election Procedure 1, will not in general reproduce the results of a party-list election when each voter votes a party list.

6.3 Election Procedure 4: Proportionality without an Electability Test

The simplest procedure that agrees with party-list elections when voters vote a party list as in Election Procedure 2, and provides no benefit to Woodall free riding as in Election Procedure 3, is presented below as Election Procedure 4. It satisfies divisor-method proportionality while not requiring that the electability load ever be calculated. However, its violation of later-no-harm/help is more severe than that of Election Procedure 2. It does not reduce to the Alternative Vote for the case of one seat.

If there are M candidates, the procedure first calculates which candidates would be elected in an $(M-1)$-seat election. The $M-1$ winners are entered in an election for $M-2$ seats and the one non-elected candidate is excluded and is assigned a final election load. The $M-2$ winners are entered in an election for $M-3$ seats and the one non-elected candidate is excluded and is assigned a final election load, etc. For an N-seat election, the count can stop when N hopeful candidates remain. Alternatively the count can be continued until all candidates have been assigned a final election load. In that case, candidates with the N lowest final election loads are the elected candidates in an N-seat election.

The method does not require the calculation of the electability load since it is guaranteed that for an election for $X-1$ seats for X candidates, the candidate with the lowest load is electable.[11] The proof that the method satisfies the divisor-method proportionality condition is as follows: The multi-round Election Procedure 3 for N seats will elect the same N candidates as Election Procedure 4 if Procedure 3 is modified so that the candidate chosen for exclusion when no candidate is electable is not the hopeful candidate with the largest election load, but instead the hopeful candidate with the largest final election load as produced from Procedure 4. Agreement with the divisor method proportionality condition follows since the condition does not depend on which candidate is excluded when no candidate is electable.

Step 1. At the start of the first round every candidate is hopeful and the seat value of every ballot is zero.

Step 2. Election load f_c for each hopeful candidate c is determined from

$$\sum_{cballots}\left(g(s, f_c) - s\right) = 1,$$

where the sum is taken over all ballots where c is the topmost hopeful candidate. If there is more than one hopeful candidate, go to Step 3a. If there is just one hopeful candidate, go to Step 3b.

Step 3a. If there is more than one hopeful candidate, elect the candidate with the lowest

[11] When all remaining candidates are tied with loads equal to the electability load none are electable and a tiebreaking procedure is needed to elect one of the candidates. But it is still the case that one does not need to calculate the electability load in this situation.

load. If candidate c has just been elected, the seat value for each ballot with seat value *s* that contributed to electing c is increased to $g(s, f_c)$. Proceed to Step 2 to begin the next stage.

Step 3b. If there is just one hopeful candidate, exclude it. Its election load becomes its final election load. If all candidates are excluded (and therefore all have been assigned final election loads), the candidates with the *N* lowest final election loads are the elected candidates in an *N*-seat election and the count is concluded. Otherwise, if there is one or more elected candidate, set all elected candidates to hopeful. Set all seat values to zero. Proceed to Step 2 to begin the next round.

For this procedure, changing the number of seats without changing ballots has no effect on final election loads. Therefore, elected candidates remain elected if the count is rerun for a larger number of seats with ballots unchanged. Also, if all voters are party loyalists so that they only rank candidates from their party, although in any order and not necessarily ranking every candidate from their party, the final election loads produced by counting each party's ballots separately will not change if the ballots are all counted together. Therefore, elected candidates remain elected if the count is rerun with ballots added for a new party and the number of seats increased until the total number of seats awarded to the previous parties is at least as large as it was previously. Lastly, if all voters are party loyalists so that they only rank candidates from their party, although in any order, and they rank *all* of the members of their party, then for d'Hondt rounding only,[12] the final election loads for a party that receives *v* votes will be $1/v$, $2/v$, $3/v$, etc. An increase (decrease) in a party's votes will decrease (increase) the party's final loads without changing the loads for other parties. Therefore, for fixed number of seats, an increase in a party's votes cannot decrease the number of seats awarded to that party and a decrease to a party's votes cannot increase the number of seats awarded to that party. However, monotonicity for the individual candidates is not guaranteed since the rank of candidates within a party can change non-monotonically as party ballots are added or removed. These properties are demonstrated by Elections 5 and 6.

Election 5 (2 seats, d'Hondt)

35 ACB
33 BAC
32 CBA

The final election loads are $f_A = 1/100$, $f_B = 3/100 = 1/33.33$, and $f_C = 2/100 = 1/50$, so that candidate A is elected to a one-seat election and candidates A and C are elected to a two-seat election. For d'Hondt rounding only, for any set of 100 ballots where each voter ranked all three candidates, the final loads are guaranteed to be 1/100, 2/100, and 3/100, although which loads candidates are assigned will depend on the ballots.

The consequences of having an additional candidate, D, with 33 votes and with ballots otherwise unchanged, can be seen in Election 6.

Election 6 (2 seats, d'Hondt)

35 ACB
33 BAC
32 CBA
33 D

The final election loads for candidates A, B, and C are unchanged. Candidate D's final election load is $f_D = 1/33$. Candidate A still wins a one-seat election and candidates A and C still win a two-seat election. For STV and all of the other election procedures described in this paper, candidates A and C are elected to a two-seat election when D voters don't vote but candidates A and B are elected when the D voters vote.

A demonstration of Election Procedure 4's violation of later-no-harm/help is provided by Election 7.

Election 7 (1 seat, d'Hondt)

35 A
33 BC
32 CA

Procedure 4 elects candidate A. However if the voters who ranked candidate A first also ranked C second, the procedure would have instead elected candidate C. This shows that voters can be harmed by ranking an additional

[12] Only for d'Hondt rounding is $NF_{0,1} = F_{N-1,N}$ for all *N*, which is required for any distribution of party ballots to produce the same final election loads.

candidate. Election 8 shows that they can also be helped.

Election 8 (1 seat, d'Hondt)

35 A
33 BC
32 CA

Procedure 4 elects candidate B. However if the voters who ranked candidate A first had ranked C second, the procedure would have elected candidate A.

6.4: Election Procedure 5: Reproducing Party-List Elections, Providing No Benefit to Woodall Free-Riding, and Reducing to Alternative Vote for One Seat.

The final election procedure presented in this paper combines: excluding candidates with the largest election load as in Procedures 1-3 to provide agreement with Alternative Vote in one-seat elections, the recalling of excluded candidates after an election as in Procedure 2 to provide agreement with party list elections when voters vote a party list, and the reassigning of seat values after an exclusion as in Procedure 3 to provide no benefit from Woodall free-riding Its violation of later-no-harm/help when the election is for more than one seat is no more severe than that of Election Procedure 2. However it does not have the properties of party separability and monotonicity with respect to the number of seats, of Election Procedure 4. The procedure temporarily re-excludes all previously excluded hopeful candidates while seat values are being reassigned.

Step 1. At the start of the first stage every candidate is hopeful and the seat value of every ballot is zero. The remaining number of seats to be filled, R, is set to N, the total number of seats to be filled. Proceed to Step 4.

Step 2. Set all elected candidates to previously elected hopeful status. Set all previously excluded hopeful candidates to temporarily excluded status. Set all seat values to zero.

Step 3. Election load f_c for each previously elected hopeful candidate c is determined from

$$\sum_{cballots}\left(g\left(s,f_c\right)-s\right)=1$$

where the sum is taken over all ballots where c is the topmost hopeful candidate. Re-elect the candidate with the lowest load. If previously elected candidate c is re-elected, the seat value for each ballot with seat value s that contributed to re-electing c is increased to $g(s, f_c)$. If not all previously elected candidates have been re-elected, proceed to Step 3 for the next re-election. Otherwise recall all temporarily excluded candidates to previously excluded hopeful status.

Step 4. The election load f_c for each hopeful candidate c is determined from

$$\sum_{cballots}\left(g\left(s,f_c\right)-s\right)=1\ ,$$

where the sum is taken over all ballots on which c is the topmost hopeful candidate and the electability load is determined from

$$\sum_{active}\left(g\left(s,f_{elect}\right)-s\right)=R+1\ ,$$

where the sum is taken over all active ballots. If at least one hopeful candidate is electable go to step 5a. If no candidates are electable, go to step 5b.

Step 5a. Set the electable candidate with the lowest election load to elected. (The count can be stopped if N candidates are elected). The next stage begins. R is reduced by 1. Set all excluded candidates to previously excluded hopeful status. Proceed to Step 2.

Step 5b. Exclude the candidate with the largest election load amongst those that are the topmost hopeful candidate on at least one ballot. Also exclude all hopeful candidates that that do not appear as the topmost hopeful candidate on any ballot. If all of the candidates excluded in this step have been previously excluded proceed to Step 4. Otherwise, proceed to Step 2. Election 9 demonstrates the procedure.

Election 9, (2 seats, d'Hondt)

13 AB
8 AC
4 DAC

Stage 1: f_A = 1/21, f_D = 1/4, f_{elect} = 3/25=1/8.33. Candidates B and C are not the topmost hopeful candidate on any ballot so their loads are not calculated. The lowest election load, f_A, is lower than the electability load so candidate A is elected. Stage 2: Each of the twenty one ballots that contributed to candidate A's election are assigned a seat value of 1/21. $F_B = (13/21+1)/13 = 1/8.0$, $f_C = (8/21 +1)/8 = 1/5.8$, $f_D = 1/4$, $f_{elect} = 1/8.33$. No candidate's election load is less than the electability load. Candidate D has the largest election load, is excluded. Reweighing: With D excluded, candidate A's load is decreased to f_D =1/25. Candidate A is re-elected. Each of the twenty five ballots that contributed to candidate A's re-election is assigned a seat value of 1/25. The loads are now $f_B = (13/25 + 1)/13 =1/8.55$, $f_C = (12/25 + 1)/12 = 1/8.11$, $f_{elect} = 1/8.33$. Candidate B is elected. In the single round procedure, candidate C is elected even though 13 voters wanted A and B and only 12 wanted A and C.

7 Conclusion

In this paper, a generalization of Woodall's QPQ procedure has been presented for assigning seats from preference ballots in multi-candidate elections, using divisor methods (d'Hondt, Sainte-Laguë. Huntington-Hill, etc.) commonly used in party-list proportional representation elections. The procedure satisfies a proportionality condition that, in general, is different from Droop proportionality. Versions of the procedure can satisfy later-no-harm/help criteria or reproduce the results of party-list elections when each voter votes a party list, but not both at the same time. I gratefully acknowledge Douglas Woodall for his very helpful comments and suggestions. All errors are my own.

8 References

[1] Balinski, M. and Young H. (2001 [1982]) *Fair Representation* (2nd. ed.), Brookings Institution. (1st ed., Yale University Press, New Haven.)

[2] d'Hondt, Victor (1878) *La Répresentation Proportionelle des Parties*. Ghent.

[3] d'Hondt, Victor (1882) *Système Pratique et Raisonné de Répresentation Proportionelle*. Brussels: C. Muquardt.

[4] Droop, Henry R. (1868) *Methods of Electing Representatives*. London: Macmillan.

[5] Ernst, Lawrence R. (1994) Apportionment Methods for the House of Representatives and the Court Challenges, *Management Science*, 40 (10), 1207-1227.

[6] Farrell, D. (2001) *Electoral Systems: a Comparative Introduction*. New York: St. Martin's Press.

[7] Hare, Thomas (1859) *On the Election of Representatives Parliamentary and Municipal: A Treatise*. London: Longmans Green.

[8] Huntington, E. (1921) A New Method of Apportionment of Representatives, *Quart. Publ. Amer. Stat. Assoc*, 17, 859-1970.

[9] Huntington, E. (1928) The Apportionment of Representatives in Congress, *Transactions of the American Mathematical Society*, 30, 85-110.

[10] Jefferson, Thomas (1904 [1792]) Opinion on Apportionment Bill, in *The Writings of Thomas Jefferson*. Washington, DC, vol. 3.

[11] Malkevitch, Joseph (2002) Apportionment, American Mathematical Society, http://www.ams.org/featurecolumn/archive/apportionII1.html (Accessed 11/02/10).

[12] Phragmén, E (1895) Proportionella Val [Proportional Elections], *Svenska Spörsmål* 25, Stockholm.

[13] Phragmén, E (1899), Till Frågan om en Proportionell Valmetod [On the Question of a Proportional Election Method], *Statsvetenskaplig Tidskrift* 2 (7), 87-95.

[14] Sainte-Laguë, André (1910) 'La Representation Proportionelle et la Méthode des Moindres Carrés', *Comptes Rendus Hébdomadaires des Séances de l'Académie des Sciences* 151, 377–8.

[15] Salmi, Olli. (2002) d'Hondt without Lists, http://groups.yahoo.com/group/election-methods-list/message/10068 (Accessed 12/15/10).

[16] Salmi, Olli. (2002) d'Hondt without Lists, http://www.mail-archive.com/election-methods-list@eskimo.com/msg08476.html (Accessed 12/15/10).

[17] Schulze, Markus. (2004) Free Riding, *Voting matters* 18, 2-8.

[18] Woodall, Douglas R. (2003) QPQ, A Quota-Preferential STV-like Election Rule, *Voting matters* 17, 1-7.

About the Author

Ross Hyman is a researcher with the American Federation of Labor and Congress of Industrial Organizations (AFL-CIO), the labor union federation in the United States. He has a Ph.D. in physics and enjoys applying the methods of physics to problems in other disciplines.